8-1-2007

The Whitehead Link on the Cubic Lattice

Elizabeth Haynes
Western Kentucky University

THE WHITEHEAD LINK ON THE CUBIC LATTICE

A Thesis
Presented to
the Faculty of the Department of Mathematics
Western Kentucky University
Bowling Green, Kentucky

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

by
Elizabeth Lydia Haynes
August, 2007

THE WHITEHEAD LINK ON THE CUBIC LATTICE

Date Recommended 7/11/07

Claus Ernst

Director of Thesis

Uta Ziegler

Bruce Kessler

Richard G. Bowker August 8, 2007

Dean, Graduate Studies and Research Date

Acknowledgements

I would like to thank God for bringing me to Western and giving me such wonderful opportunities here.

I would like to thank my family for their love and support. I send a big thanks to my mom, who tried her best to understand what I was talking about when we discussed my thesis. I also thank my godparents for their unwavering support.

To my advisor, Dr. Claus Ernst: thank you for your dedication and patience. You always made time to answer my questions and you encouraged me so much. It was a pleasure to have you for an advisor.

Thanks to all the Mathematics Department faculty and staff. I have learned so much in the past two years (more than just the contents of this thesis). Your enthusiasm for teaching and learning has been passed on to me, and I look forward to passing that on to others.

I owe more than words can say to my fellow graduate students. You tried to ensure I did not become too stressed, but gave me a shoulder to cry on when I did anyway. You encouraged me when I was stuck, and celebrated with me every breakthrough I had. Thank you for everything.

Table of Contents

THE WHITEHEAD LINK ON THE CUBIC LATTICE

Name: Elizabeth Lydia Haynes Date: August, 2007 Pages: 79

Directed by Dr. Claus Ernst
Department of Mathematics Western Kentucky University

Abstract

The cubic lattice is a graph in $\mathfrak{R}^3$ where the vertices are points with integer coordinates and edges are unit length line segments parallel to the x-, y-, or z-axis. A step is a line segment that connects one vertex to a neighboring vertex one unit away in the x-, y-, or z-direction. This thesis will show that the Whitehead Link needs at least 34 steps to be embedded on the cubic lattice.

CHAPTER ONE

Introduction and Definitions

This thesis will show the Whitehead link cannot be formed on the cubic lattice with fewer than 34 steps. The following definitions will be used to achieve this purpose.

Definition 1.1: A knot is a closed curve in $\Re^3$ that does not intersect itself [1]. A knot that is topologically equivalent to a planar circle, i.e. can be deformed to a planar circle without breaking the strands is known as the trivial knot or as the unknot (see also Definition 1.6).

Definition 1.2: A link is a finite union of knots that do not intersect. An individual knot of the link is called a component. A split link is a link where one of the components can be pulled an arbitrarily large distance from the other components without stretching any of the other components. A trivial link (or unlink) is a finite collection of unknots, where any two of the components form a split link. A non-split link is a link that is not split.

Definition 1.3: A projection of a link L is the image f(L), of a continuous function $f : \Re^3 \rightarrow \Re^2$.

Definition 1.4: A projection is called a regular projection if the following conditions hold:

1. No more than two arcs of the link project to one intersection.
2. There are only finitely many intersections.
3. There exists an $\varepsilon > 0$ such that moving any arc by distance ε in the projection plane does not reduce the number of intersections [1].

The left of Figure 1.1 shows two examples of where the above conditions were violated and the right of Figure 1.1 shows what is acceptable.

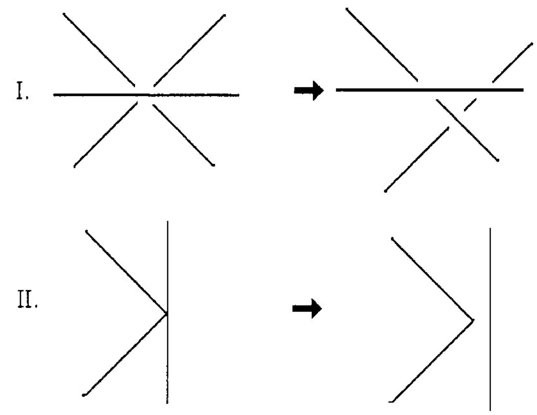

Figure 1.1

Fact 1.1: Every knot and link has a regular projection [2]. It is customary to disconnect one of the arcs at an intersection to show which arc passes over the other.

Definition 1.5: <u>Reidermeister moves</u> are deformations on a regular projection of a link that will not change the topological structures of the knot or link. There are three types of Reidermeister moves, which are shown in Figure 1.2. It is important to note that the mirror images of these moves are also possible.

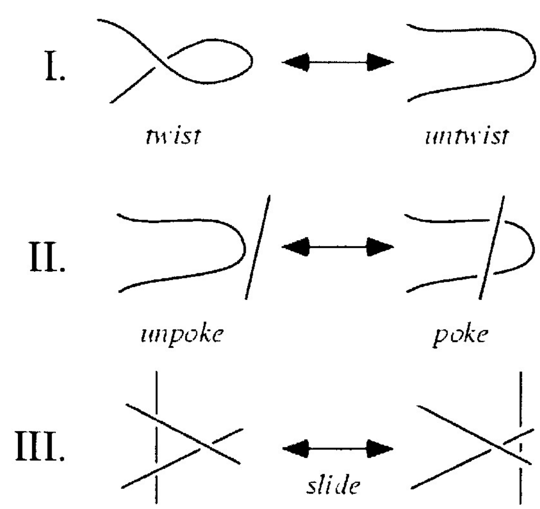

Figure 1.2

Definition 1.6: Let A and B be topological spaces. Then A is topologically equivalent (or homeomorphic) to B if there exists a continuous invertible function $f : A \to B$ (a homeomorphism) with continuous inverse $f^{-1} : B \to A$. [3]

Definition 1.7: An isotopy is an orientation preserving piecewise linear homeomorphism. Links L_1 and L_2 are equivalent if there is an isotopy $h : R^3 \to R^3$ such that $h(L_1) = h(L_2)$. [4]

Definition 1.8: A topological invariant is a criterium α such that whenever X and Y are topologically equivalent, then α(X) = α(Y).

Theorem 1.1: Given two regular projections of links J and K, if one projection can be transformed into the other by a finite sequence of Reidermeister moves and plane isotopies, then the links J and K are topologically equivalent. Conversely, any two regular projections of two topologically equivalent links are related by a finite sequence of Reidermeister moves [1, 2, 4].

Definition 1.9: Given a link L with regular projection D, and an orientation of each component. Assign a sign of ±1 to each crossing of D as follows: For each crossing, take the overpass and rotate it until the overpass is on top of the underpass such that both are oriented the same direction. If this is accomplished by a counterclockwise rotation (clockwise rotation), give the crossing a sign of positive one (negative one). See Figure 1.3. The linking number of an oriented link with two components (C_1 and C_2) is defined as

$$lk(C_1, C_2) = \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i$$

where ε_i is the ± 1 of the i^{th} crossing between components C_1 and C_2 in the projection D and n is the total number of crossings between components C_1 and C_2.

Note that n will always be even, since two simple closed curves in the plane have an even number of intersections if they are in general position. A sum of an even number of $\pm$ 1's is always even, thus the sum $\sum_{i=1}^{n} \varepsilon_i$ is always divisible by two.

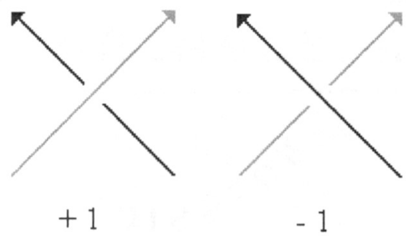

Figure 1.3

Theorem 1.2: The linking number of a link is a topological invariant of two component links [4].

Example 1.1: The trivial link, shown in Figure 1.4 (a) has linking number 0. The Hopf link, shown in Figure 1.4 (b) has linking number 1.

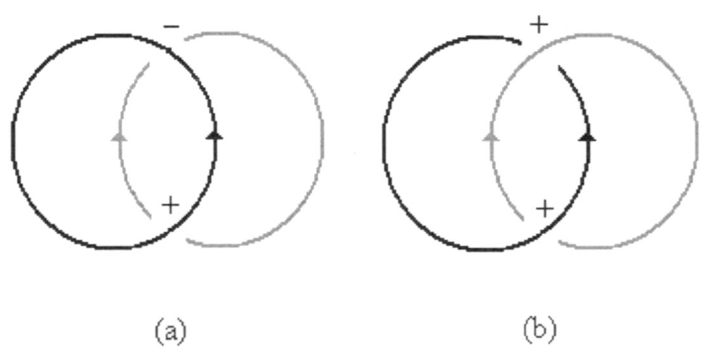

Figure 1.4

It is important to remember that while all trivial links have a linking number of zero, the converse is not true. For example, the Whitehead link (Figure 1.5) has a linking number of zero, but it is a non-trivial link.

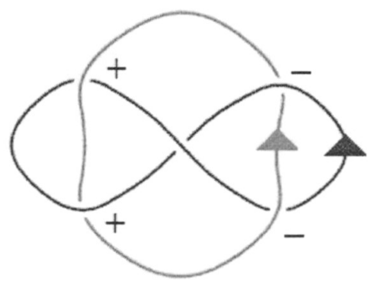

Figure 1.5

Definition 1.10: The <u>cubic lattice</u> is a graph in $\Re^3$ where the vertices are the points with integer coordinates and edges are unit length line segments parallel to the x-, y-, or z-axis. Unless otherwise noted, the standard coordinate system we will use in all figures is shown in Figure 1.6, that is, the z-axis is vertical on the page, the x-axis is perpendicular to the page, and the y-axis is horizontal on the page.

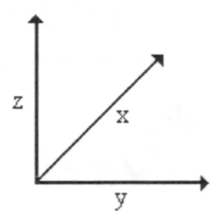

Figure 1.6

Definition 1.11: A <u>lattice link</u> is an embedding of a link onto the cubic lattice where:

1. Each component is a simple closed lattice curve.
2. No two components intersect.

In the future, "simple" and "non-intersecting" is understood when we discuss curves and links on the lattice.

Definition 1.12: A step is a line segment that connects one vertex to a neighboring vertex one unit away in the x-, y-, or z-direction. An x-step is a step parallel to the x-axis. Likewise, a y-step (or z-step) is a step parallel to the y-axis (or z-axis). The length of an embedded simple closed lattice curve is the total number of steps used in the embedding.

Fact 1.2: Any closed curve on the lattice has even length.

Pick a starting vertex v. If we move *a* x-steps away from the x-coordinate of v, then we must move *a* x-steps towards that x-coordinate to obtain a closed curve. Thus the number of x-steps must be even. The same argument can be made for the number of y-steps and z-steps. Thus a closed component requires an even number of steps (actually the length is the sum of three even non-negative integers) and the smallest such component has length 4, as shown in Figure 1.7.

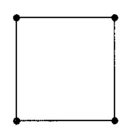

Figure 1.7

Definition 1.13: A lattice disk is a union of lattice squares that is topologically a two-dimensional disk.

Definition 1.14: A strand from vertices a to b is a line segment $\overline{ab}$ on the lattice that contains at least two steps.

Definition 1.15: Let C_1 be a closed lattice curve and let D_1 be a lattice disk where $\partial(D_1) = C_1$. We say C_1 bounds a vertex *v* (see Figure 1.8) if there exists a lattice vertex *v* in D_1 (*v* not an element of C_1) that will allow another closed lattice curve (C_2) to form a non-split link with C_1 through the vertex *v*.

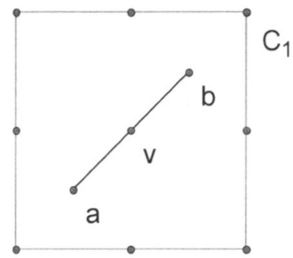

Figure 1.8

Definition 1.16: Let C_1 be a closed lattice curve and let D_1 be a lattice disk where the boundary of D_1 is C_1 ($\partial(D_1) = C_1$). We say a path P passes through C_1 if there exists a lattice vertex v such that $v \notin C_1, v \in D_1, v \in P$ that will allow another closed lattice curve (C_2) to form a non-split link with C_1 through the vertex *v*. In Figure 1.8, $\overline{ab}$ passes through C_1.

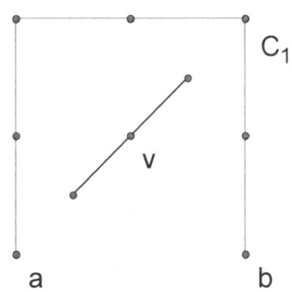

Figure 1.9

Definition 1.17: Given a lattice path P, with endpoints *a*, *b*, where *a* and *b* are on a line parallel to the x-, y-, or z-axis, and $P \cap \overline{ab} = \{a, b\}$. Let $C = P \cup \overline{ab}$. If C bounds a vertex *v*, then the path P bounds a vertex v (see Figure 1.9).

Definition 1.18: A minimal representation of a link L on the cubic lattice is a representation of L on the lattice that contains the fewest possible number of total steps. The minimal length of L on the lattice is the total number of steps in a minimal representation. For example, the Hopf link (Figure 1.4 (b)) has a minimal length of 16, as shown in Figure 1.10.

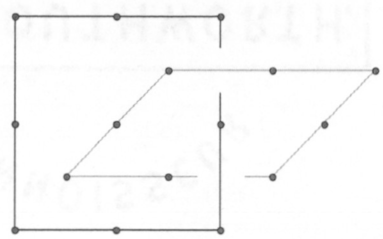

Figure 1.10

Figure 1.11 shows one minimal representation of The Whitehead Link. That the lattice link in Figure 1.11 (b) is the same as the smooth Whitehead link in Figure 1.5 can be seen by an isotopy as in Figure 1.12.

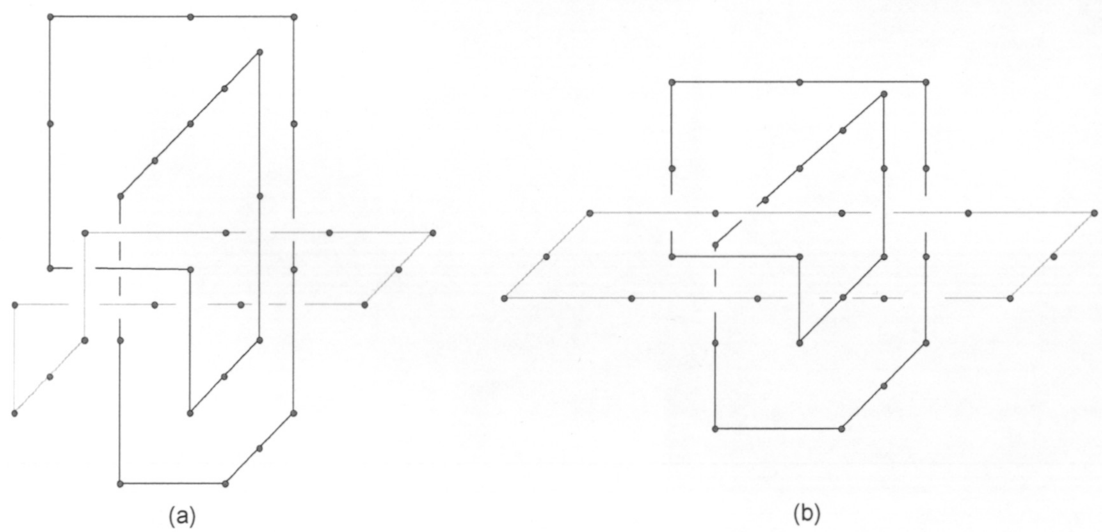

Figure 1.11

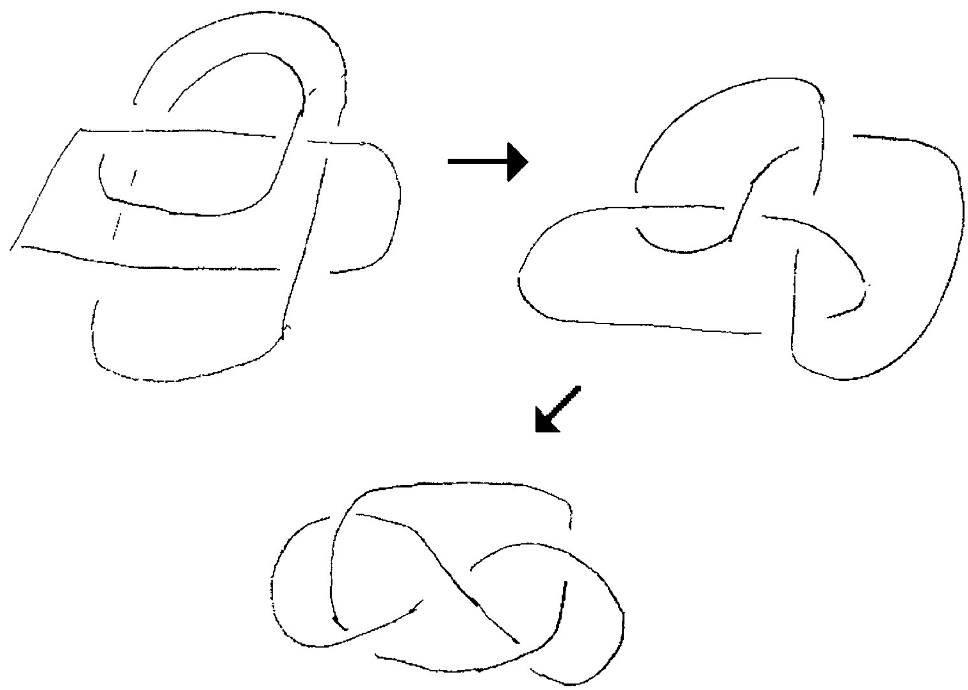

Figure 1.12

Definition 1.19: A component C_1 on the cubic lattice is called <u>reducible</u> if there exists a shorter component C_2 topologically equivalent to C_1 on the cubic lattice.

The above definition implies any curve using 3 sides of a lattice square is reducible.

Definition 1.20: Given a path P_1 with endpoints a and b on the cubic lattice, a <u>square move</u> is a path P_2 (with endpoints a and b) where $P_1 \cup P_2$ forms a unit lattice square. Figure 1.13 (a) shows one square move, Figure 1.13 (b) shows the change to the path resulting from two consecutive square moves.

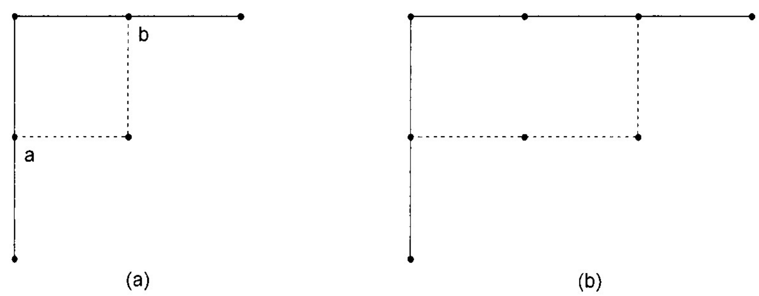

Figure 1.13

Note that on the Whitehead link in Figure 1.11, several square moves are possible to generate different minimal lattice representations of the Whitehead link.

CHAPTER TWO

The Whitehead Link

The main theorem of this thesis is the following:

Theorem 2.1: The Whitehead link has minimal length of 34.

Figure 1.11 shows a representation of The Whitehead link with length 34. In this thesis, we will systematically eliminate small lattice representation of two curves to show that it is impossible to form the Whitehead link on the cubic lattice with less than 34 steps.

The Whitehead Link has two components, C_1 and C_2. Since the components are closed curves, each component must have an even integer length. Without loss of generality, assume C_1 is the shorter component. It takes at least 4 steps to create a closed curve, so we must consider the cases where length(C_1) = {4, 6, 8, 10, 12, 14, 16}. Since the Whitehead Link is a non-trivial link, each component must bound a vertex for a strand of the other component to pass through.

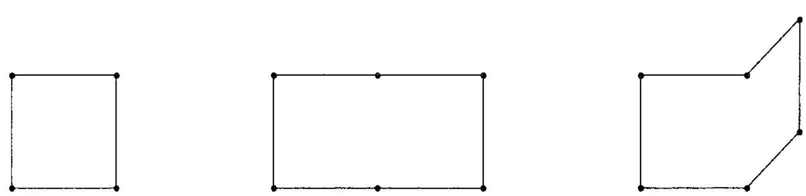

Figure 2.1

Figure 2.1 shows components of length 4 and 6. No component of length 4 or 6 can bound a vertex. There is only one configuration of length 8 that will have such a vertex, shown in Figure 2.2.

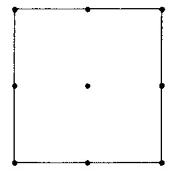

Figure 2.2

The linking number of the Whitehead Link is zero. In order for this to occur, each component must pass through the other at least twice, with half of the strands oriented opposite to the other half.

A component of length 8 can only allow one strand can pass through it. Thus each component must be at least 10 units long.

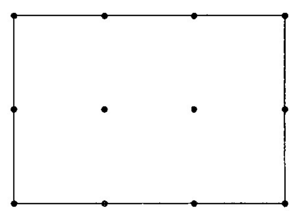

Figure 2.3

It is easy to see that any 10 step non-planar configuration cannot bound two vertices. Out of all the possible 10 step configurations, Figure 2.3 shows the only one that will allow two strands to pass through. Let C_1 be one of the components of the Whitehead link on the lattice. At this point, we know that the length of C_1 is at least 10. We will now consider the case length(C_1) = 10 in more detail.

Case 1: C_1 has length 10

Without loss of generality, let C_1 use 4 x-steps and 6 y-steps. Figure 2.4 shows a projection into the y-z plane.

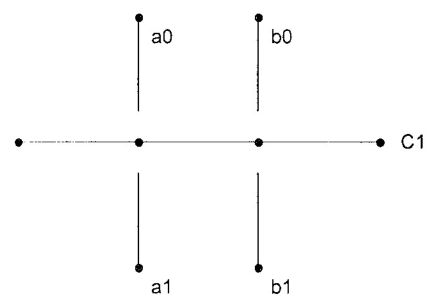

Figure 2.4

C_1 forces the four z-steps for C_2 as shown in Figure 2.4. Pick an orientation on C_2; this induces an orientation on the two strands $\overline{a_0a_1}$ and $\overline{b_0b_1}$. The strands must be oriented in opposite directions to achieve a linking number of zero, so without loss of generality, we assume a_0 must connect to b_0 and b_1 must connect to a_1. A path that connects a_0 to b_0 would be a closed component if we included the line segment $\overline{a_0b_0}$. Thus it will take an odd number of steps to connect a_0 to b_0. By a similar argument, it will take an odd number of steps to connect a_1 to b_1. Let P_1 be the path that connects a_0 to b_0 and let P_2 be the path that connects a_1 to b_1. Using symmetry, we can assume that without loss of generality, the length of P_1 is less or equal to the length of P_2. In order for the Whitehead link to have a length of no more than 34 steps, P_1 must have length less than 10 steps (since the configuration in Figure 2.4 already uses 14 steps).

The Whitehead link is a non-trivial link, so there must be something that prohibits P_1 from retracting through C_1, i.e. the path from a_0 to b_0 must bound a vertex that the other path, P_2, passes through. A closed component must have at least length 8 for this to occur, so this path must have at least length 7. We will now consider the two cases where: 1) length(P_1) = 7, and 2) length(P_1) = 9.

Case 1.1: P_1 has length 7

Using seven steps to connect them, there is only one configuration (up to symmetry) that will bound a vertex, this configuration is shown in Figure 2.5.

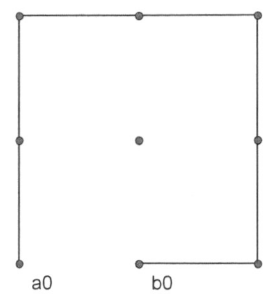

Figure 2.5

This path can be joined to the configuration in Figure 2.4 so that it is in the same plane as $\overline{a_0 a_1}$ and $\overline{b_0 b_1}$, or perpendicular to that plane.

Case 1.1a: Consider the case where P_1 is perpendicular to the line segments $\overline{a_0 a_1}$ and $\overline{b_0 b_1}$, one such case is shown in Figure 2.6.

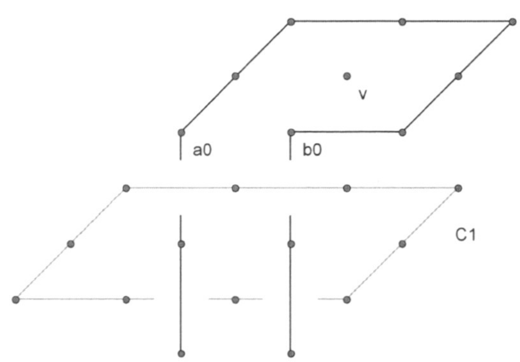

Figure 2.6

If C_2 does not occupy vertex v, then C_1 is reducible. However, C_2 cannot pass through vertex v and remain disjoint from C_1, a contradiction. Thus, this configuration of P_1 is impossible. Note that there are three other positions of P_1, shown in Figure 2.7, that do not have to be considered because they are symmetric to the given situation. In many future instances, this will happen and we will restrict ourselves to only one of these cases.

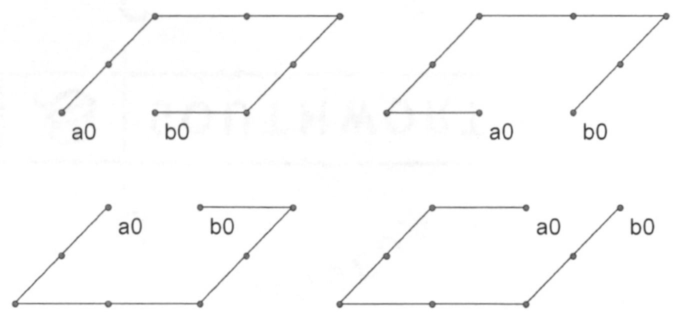

Figure 2.7

Case 1.1b: Consider the case where P_1 and the line segments $\overline{a_0a_1}$ and $\overline{b_0b_1}$ are planar, as shown in Figure 2.8. Again, there are cases symmetric to the given one. We will discuss this case in detail to give the flavor of the type of arguments we need to use. In future cases, some of these details will be omitted.

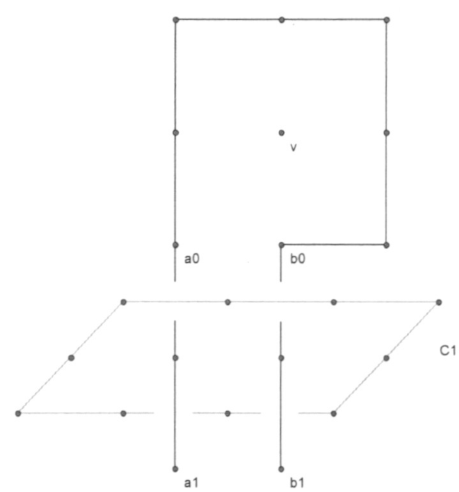

Figure 2.8

Since C_1 cannot be reducible, the vertex v must be a vertex of C_2. We need to connect a_1 to v and v to b_1. Let $P_2 = Q_1 \cup Q_2$, where Q_1 is the path that connects a_1 to v and Q_2 is the path that connects v to b_1. There are many possibilities for Q_1 and Q_2 and will not discuss possibilities that are clearly not minimal.

Case 1.1b (1): Starting at a_1, the first step of Q_1 could be a z-step. Since our configuration already uses the z-steps that connect a_0 to a_1, using a z-step for our first step of Q_1 can only move us away from v, so it cannot yield a minimal path.

Case 1.1b (2): Starting at a_1, the first step of Q_1 could be a y-step. It cannot be the y-step that connects a_1 to b_1 or else we would close component C_2 and form a split-link. Thus this y-step must look like the one if Figure 2.9.

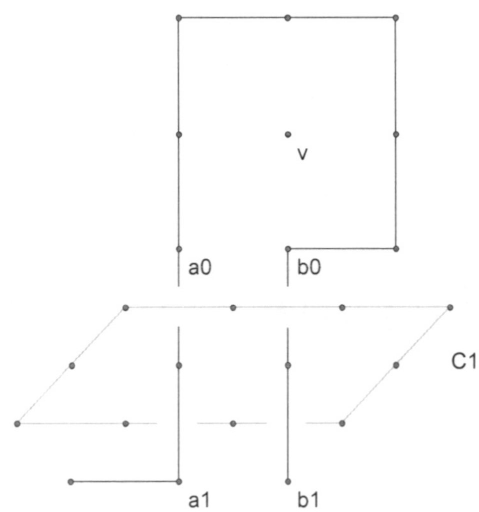

Figure 2.9

Moving one y-step does not allow us to avoid C_1, so we must move another y-step to do so. Then we need to move 3 z-steps to approach v. Now we are 3 y-steps away from v, but we must avoid P_1. The simplest way to avoid P_1 is to move one x-step and another x-step at the end of Q_1. This means Q_1 has length 10. Figure 2.10 shows some examples for Q_1. The dark bolded line shows the steps in the order listed. The lighter bolded line shows these same steps, moving the opposite x-steps. The dashed lines show how we could make these x-steps at almost any vertex of P_1. The thin line indicates a square move that is possible. All the paths shown in Figure 2.10 have a length of 10. This gives a total length of 14 + 7 + 10 = 31. Clearly, it is not possible to connect b_1 to v in 3 steps. (Actually Q_2 must have at least 7 steps, resulting in configurations of the Whitehead link using 38 steps.)

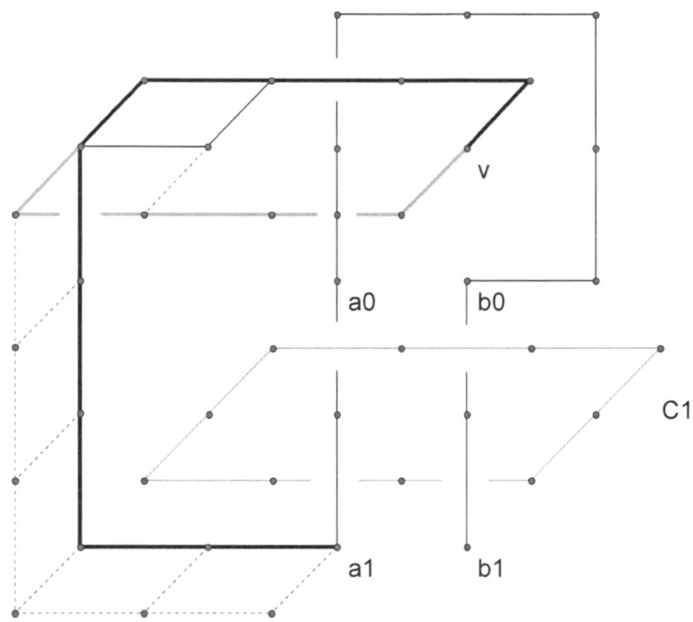

Figure 2.10

Case 1.1b (3): Starting at a_1, the first step of Q_1 could be an x-step. Moving one x-step does not let us avoid C_1, so we must move another x-step. We also need 3 z-steps to approach v. From there, we need to move one y-step and 2 x-steps to reach v, and we can do so in any order so long as we avoid P_1. This configuration of Q_1 requires 8 steps, so a minimal Q_1 contains 4 x-steps, 1 y-step and 3 z-steps. Figure 2.11 shows some examples for Q_1.

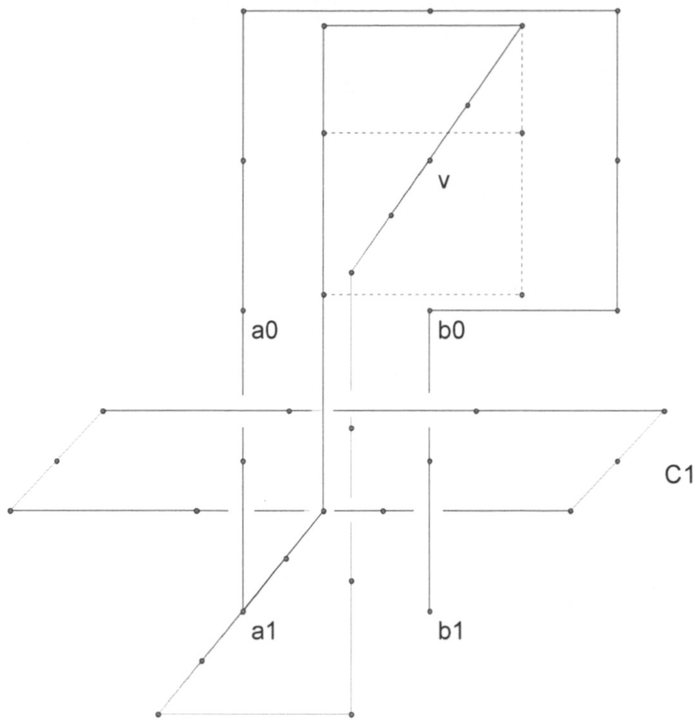

Figure 2.11

Q_2 must avoid component C_1 and paths P_1 and Q_1. By repeating the same process as we did for Q1, it can easily be seen that a minimal Q_2 has 7 steps and contains 4 x-steps and 3 z-steps. The three z-steps are clearly necessary. The four x-steps are the smallest number of possible steps that allow us to avoid C_1. This means the configuration has 36 steps total, so it is not minimal. Figure 2.12 shows one such lattice diagram. Note that this configuration of 36 steps is by no means unique. There are square moves possible, as seen in Figure 2.10 and Figure 2.11. However, in all these configurations, a P_1 of seven steps forces P_2 to be 15 steps long, giving a total length of 36.

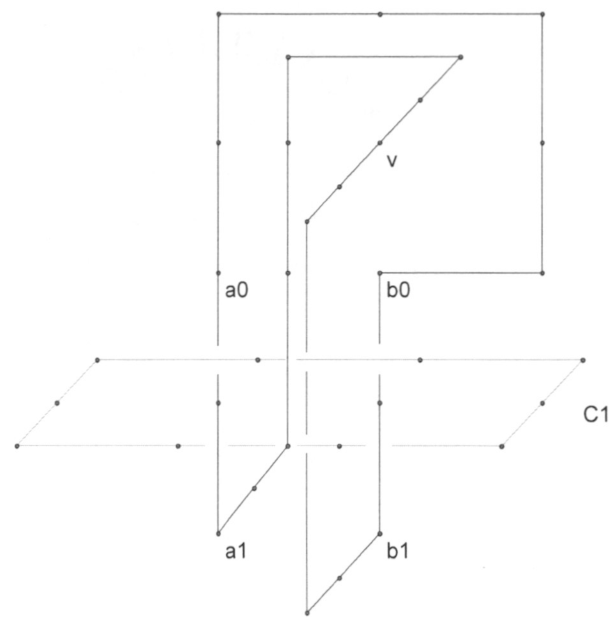

Figure 2.12

This shows we cannot have a minimal diagram when P_1 has length 7. Next, we will consider the case when length(P_1) = 9.

Case 1.2: P_1 has length 9

The path P_1 could have length 9. Since $P_1 \cup \overline{a_0 b_0}$ is a component of length 10, and $\overline{a_0 b_0}$ is a y-step, we have the following cases to consider:

Case	x-steps	y-steps	z-steps
1	0	1	8
2	0	3	6
3	0	5	4
4	0	7	2
5	2	1	6
6	2	3	4
7	2	5	2
8	2	7	0
9	4	1	4
10	4	3	2
11	4	5	0
12	6	1	2
13	6	3	0
14	8	1	0

Table 2.1

Case 1.2.1: Figure 2.13 shows the only possible configuration where P_1 has 0 x-steps, 1 y-step, and 8 z-steps. P_1 is reducible and does not bound a vertex, so this case is impossible.

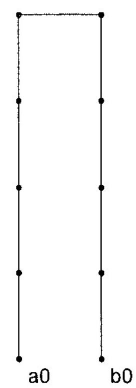

Figure 2.13

Case 1.2.2: Figure 2.14 shows the possible configurations of P_1 up to symmetry, where P_1 has 0 x-steps, 3 y-steps, and 6 z-steps.

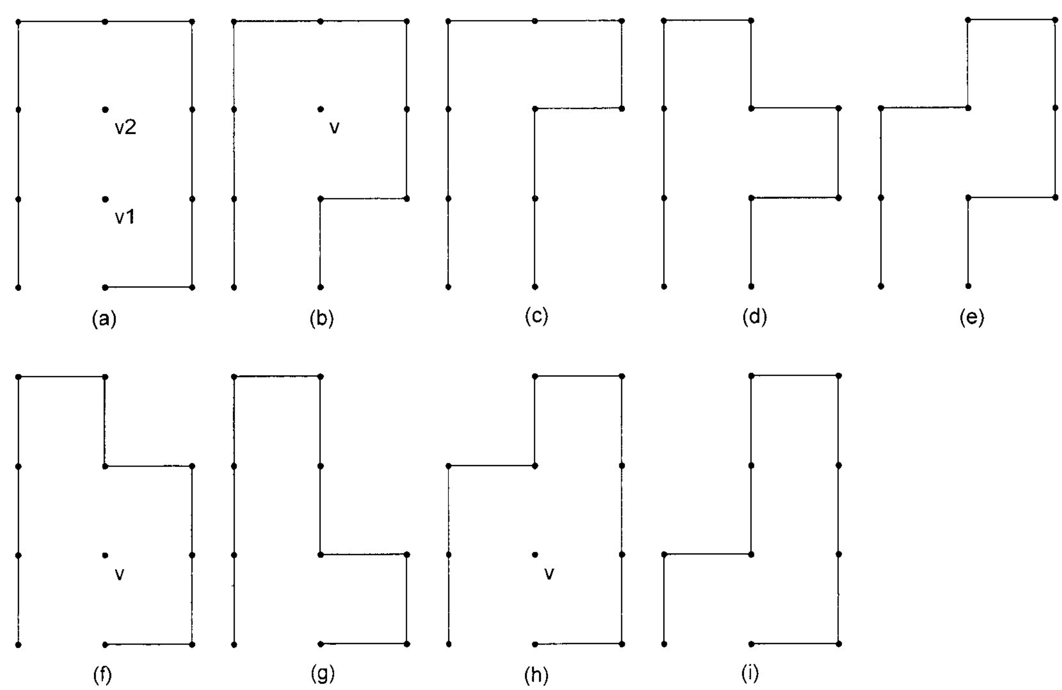

Figure 2.14

Configurations (c), (d), (e), (g), and (i) do not bound a vertex. Configurations (f) and (h) are reducible. Without loss of generality, we will only discuss configurations (a) and (b) and not the cases symmetric to the ones shown in Figure 2.14.

Case 1.2.2a: Consider P_1 to be the configuration shown in Figure 2.14 (a). If P_2 passes through v_1 and not v_2, then P_1 is clearly reducible to the P_1 in Case 1.1b, which we already described. If P_2 passes through v_2 (as shown in Figure 2.15), then Q_1 will require one more z-step than the Q_1 of Case 1.1b. Q_2 will also require an additional z-step, thus this gives us a total of 40 steps, which is not minimal.

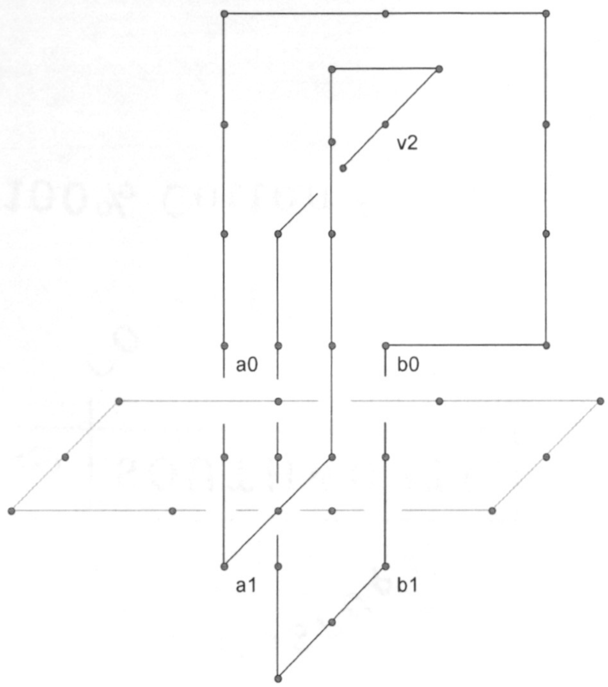

Figure 2.15

Case 1.2.2b: Consider P_1 to be the configuration shown in Figure 2.14 (b). This is very similar to utilizing v_2 in Case 1.2.2a, so it will take 40 steps, which is not minimal.

Case 1.2.3: Figure 2.16 shows the possible configurations of P_1 with respect to symmetry, where P_1 has 0 x-steps, 5 y-steps, and 4 z-steps.

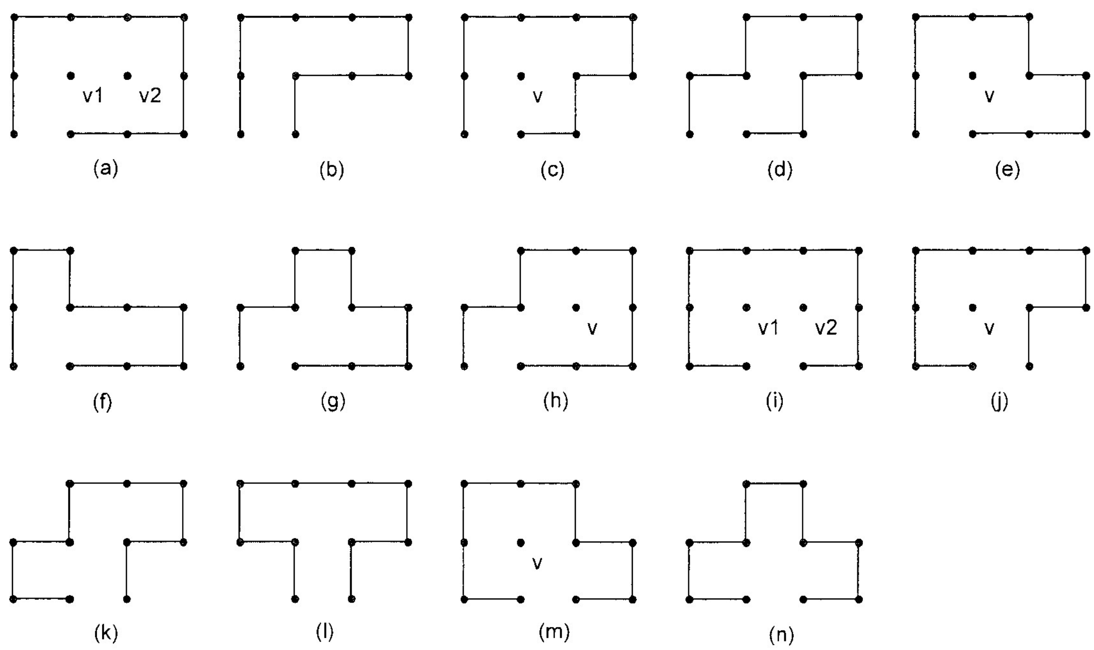

Figure 2.16

Configurations (b), (d), (f), (g), (k), (l), and (n) do not bound a vertex and are reducible, so we can dismiss these cases. In addition, configurations (c), (e), (j), and (m) are reducible, so we can dismiss these cases as well. This leaves only the cases (a), (h) and (i).

Case 1.2.3a: Consider P_1 to be the configuration shown in Figure 2.16 (a). Consider P_2. If P_2 passes through v_1 and not v_2, then the path P_1 is reducible to the P_1 in Case 1.1b. If P_2 passes through v_2 (shown in Figure 2.17), then Q_1 and Q_2 will each require one more y-step than the Q_1 and Q_2 of Case 1.1b, giving us a total of 40 steps (the details are left to the reader). Thus it is not minimal.

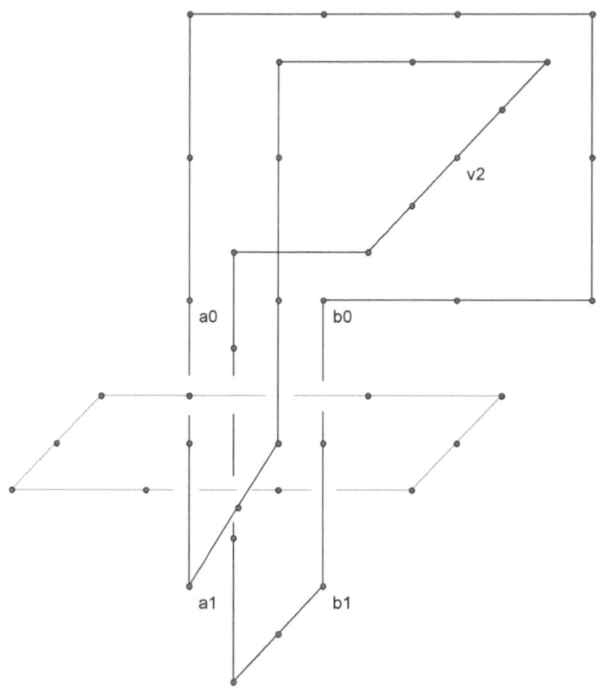

Figure 2.17

Case 1.2.3h: Consider P_1 to be the configuration shown in Figure 2.16 (h). This is very similar to P_2 passing through v_2 of Case 1.2.3a. Thus this configuration requires 40 steps, so it is not minimal.

Case 1.2.3i: Consider P_1 to be the configuration shown in Figure 2.16 (i). If P_2 passes through only v_1 or v_2 then P_1 is reducible to Case 1.1b or its mirror image respectively, so it is not minimal. If P_2 contains both v_1 and v_2, then for P_2 to have any chance of a reasonably small length it must contain the edge $\overline{v_1 v_2}$. In this case, it will take a minimum of 4 x-steps and 3 z-steps to connect a_1 to v_1, 4 x-steps and 3 z-steps to connect b_1 to v_2, and 1 y-step to connect v_1 to v_2. Thus P_2 will have length 15 and this configuration will have a total length of 38, which is not minimal.

Case 1.2.4: Any P_1 with 0 x-steps, 7 y-steps, and 2 z-steps cannot bound a vertex, so we can dismiss this case.

Case 1.2.5: Any path with 0 x-steps, 1 y-step, and 6 z-steps does not bound a vertex. Adding 2 x-steps to such a path does not alter the path enough to allow it to bound a vertex. Thus, P_1 cannot have 2 x-steps, 1 y-step, and 6 z-steps.

Case 1.2.6: With respect to symmetry, any P_1 with 2 x-steps, 3 y-steps, and 4 z-steps must lie on the "ribbon" shown in Figure 2.18.

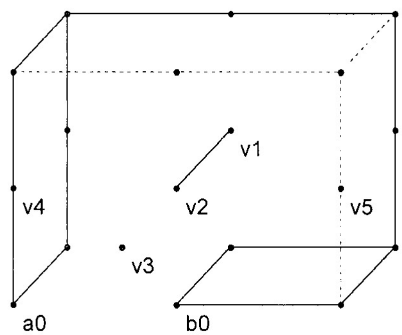

Figure 2.18

P_2 must contain $\overline{v_1v_2}$, and then we have three sub-cases: 1) $\overline{v_2v_3} \in P_2$, 2) $\overline{v_2v_4} \in P_2$, or 3) $\overline{v_2v_5} \in P_2$.

Case 1.2.6 (1): Given $\overline{v_2v_3} \in P_2$, then P_1 is reducible to the P_1 in Case 1.1, so it is not minimal.

Case 1.2.6 (2): Given $\overline{v_2v_4} \in P_2$, then it will take at least 7 steps to connect a_1 to v_4 (0 x-steps, 4 y-steps, and 3 z-steps). To connect b_1 to v_1 will require 6 steps (3 x-steps, 0 y-steps, and 3 z-steps). Thus this configuration will have length 38, which is not minimal.

Case 1.2.6 (3): Given $\overline{v_2v_5} \in P_2$, then to connect a_1 to v_1 will require 7 steps (3 x-steps, 1 y-step, and 3 z-steps). To connect b_1 to v_5 will require 6 steps (0 x-steps, 3 y-steps, and 3 z-steps). Thus this configuration will have length 38, which is not minimal.

Case 1.2.7: Consider P_1 to have 2 x-steps, 5 y-steps, and 2 z-steps. When any such path is projected to the x-z plane, we see one of two cases, as shown in Figure 2.19.

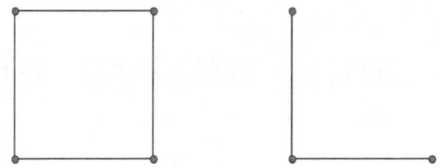

Figure 2.19

Neither of these cases can bound a vertex, so no P_1 with 2 x-steps, 5 y-steps, and 2 z-steps will bound a vertex.

Case 1.2.8: No path with 2 x-steps, 7 y-steps, and 0 x-steps can bound a vertex.

Case 1.2.9: Given 4 x-steps, 1 y-step, and 4 z-steps, then P_1 must lie on the "ribbon" as shown in Figure 2.20. Also $\overline{v_1 v_2} \in P_2$.

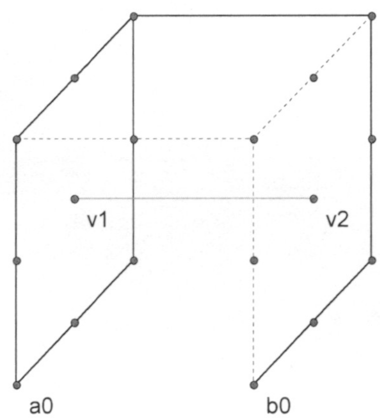

Figure 2.20

To connect a_1 to v_1 will require at least 6 steps (3 x-step, 0 y-steps and 3 z-steps). To connect b_1 to v_2 will also need at least 6 steps (3 x-step, 0 y-steps and 3 z-steps). Thus this configuration contains 36 steps, which is not minimal.

Case 1.2.10: Given 4 x-steps, 3 y-steps, and 2 z-steps, then P_1 must lie on the "ribbon" as shown in Figure 2.21. $\overline{v_1v_2} \in P_2$ and we have six cases: 1) $\overline{v_1v_3}, \overline{v_2v_7} \in P_2$, 2) $\overline{v_1v_5}, \overline{v_2v_7} \in P_2$, 3) $\overline{v_1v_4}, \overline{v_2v_6} \in P_2$, 4) $\overline{v_1v_4}, \overline{v_2v_8} \in P_2$, 5) $\overline{v_1v_9}, \overline{v_2v_6} \in P_2$, or 6) $\overline{v_1v_9}, \overline{v_2v_8} \in P_2$. These cases are very similar; we will discuss the first and leave the rest to the reader.

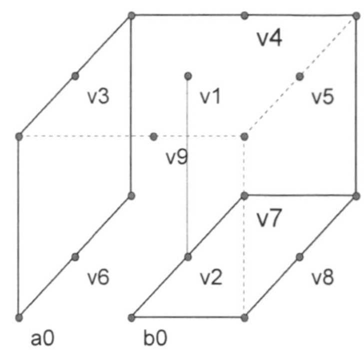

Figure 2.21

Given $\overline{v_1v_3}, \overline{v_2v_7} \in P_2$, it will take at least 8 steps to connect a_1 to v_3 (1 x-step, 4 y-steps, and 3 z-steps or 5 x-steps, 0 y-steps, and 3 z-steps). Three steps are the minimum for connecting b_1 to v_7 (2 x-steps, 0 y-steps, and 2 z-steps). Hence this configuration will give us 38 steps total, which is not minimal.

Case 1.2.11: Similar to Case 1.2.3, with respect to symmetry, there are six possibilities for P_1 to bound a vertex given 4 x-steps, 5 y-steps, and 0 z-steps, but only three that are not reducible. These 3 are shown in Figure 2.22. Due to the position of C_1, none of these three bound a vertex that can be used by P_2.

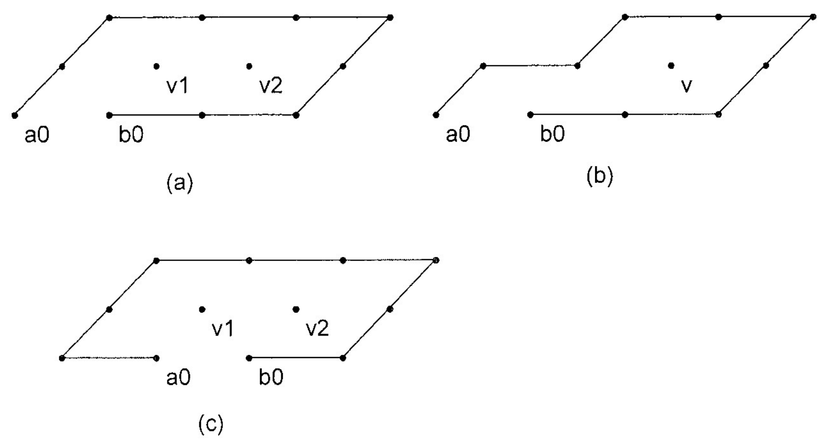

Figure 2.22

Case 1.2.12: No path of 6 x-steps, 1 y-step, and 2 z-steps can bound a vertex. (See Case 1.2.5)

Case 1.2.13: Similar to Case 1.2.2, there are two possibilities of a path with 6 x-steps, 3 y-steps and 0 z-steps, that are not reducible, to bound a vertex. These two are listed in Figure 2.23.

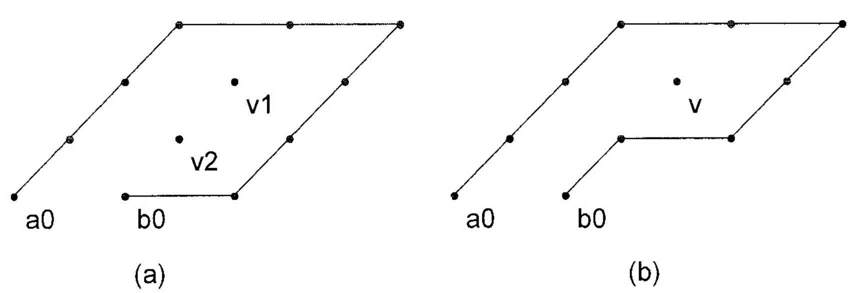

Figure 2.23

If P_2 does not pass through v_1 of Figure 2.23 (a), then P_1 is reducible. Thus Figure 2.23 (a) and (b) are very similar so we will only discuss one case.

Consider P_1 as shown in Figure 2.23 (a). Starting at a_1, we can move 2 x-steps, then 1 y-step, then 3 z-steps, then 2 y-steps and finally 1 more z-step to connect to v_1.

Starting at b_1, we can move 2 x-steps and then 2 z-steps to connect to v_1. One such Q_1 and Q_2 are shown in Figure 2.24.

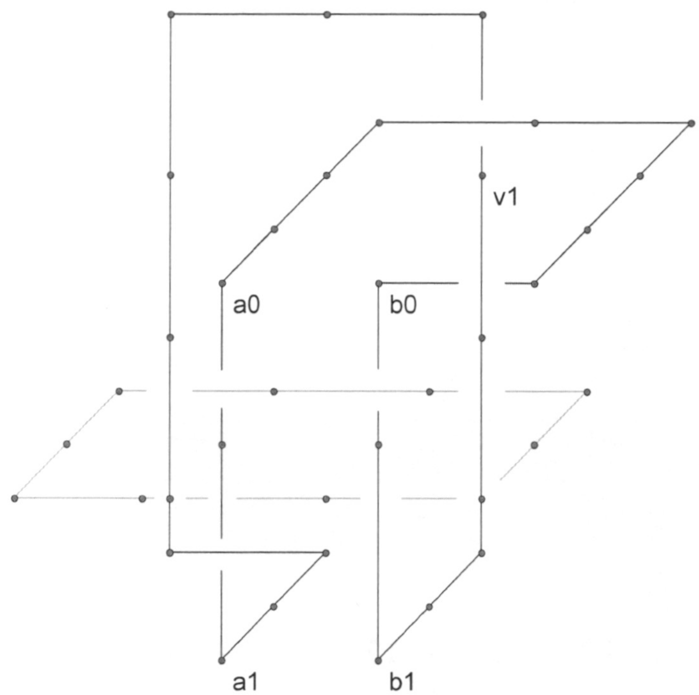

Figure 2.24

Thus we have a total length of 36, which is not minimal.

Case 1.2.14: No path with only 8 x-steps and 1 y-step can bound a vertex. (See Case 1.2.1.)

This proves that a minimal representation of the Whitehead Link cannot be formed when one component has length 10. Therefore, we assume one component has a length greater than or equal to 12.

Case 2: C_1 has length 12

In Case 1, C_1 had to be planar to allow two strands to pass through. In Case 2, C_1 can be planar or non-planar; we will first address the cases where C_1 is planar. With respect to symmetry, there are 4 cases where C_1 is planar, bounds at least two vertices, and is not immediately reducible. These 4 cases are shown in Figure 2.25.

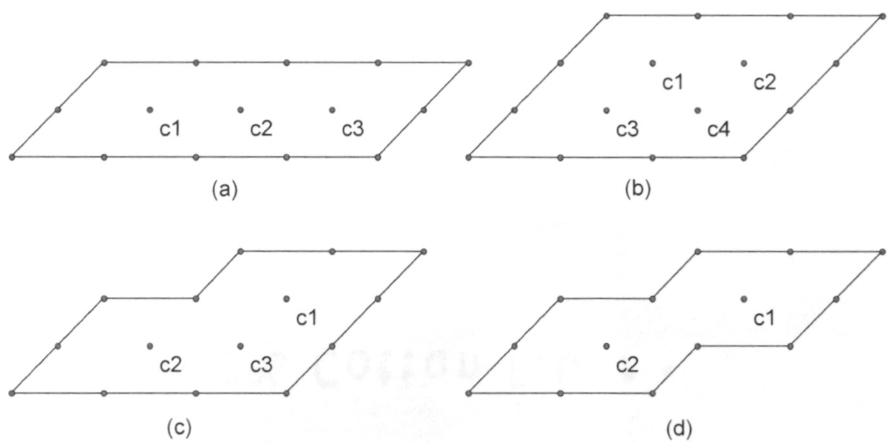

Figure 2.25

Case 2.1: Consider C_1 to be as shown in Figure 2.25 (a). If 3 strands pass through C_1, then the configuration cannot have a linking number of zero, a contradiction. Thus only two strands can pass through C_1. If the two strands use only vertices c_1 and c_2 (or c_2 and c_3) then C_1 is reducible to length 10 and is not minimal. Thus C_2 must use c_1 and c_3; we will first consider the case where the two strands use all three vertices, as shown in Figure 2.26. Again, up to symmetry, this is the only case we need to consider.

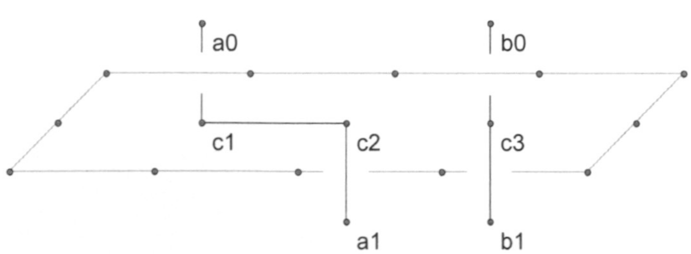

Figure 2.26

Since a_0 and b_0 are separated by 2 y-steps, P_1 must have an even number of steps. The smallest possible P_1 that will bound a vertex has length 4 (see Definition 1.17, using the path $P = P_1 \cup \overline{a_0c_1} \cup \overline{b_0c_3}$ to satisfy the definition of bound). Furthermore,

$P_1 \leq \dfrac{34-12-5}{2} = 8.5$, so we have three sub-cases: P_1 has length 4, 6, or 8.

Case 2.1.1: We can connect a_0 to b_0 using 2 x-steps and 2 y-steps, but this does not bound a vertex. Thus, a four step P_1 must contain 2 y-steps and 2 z-steps, as shown in Figure 2.27.

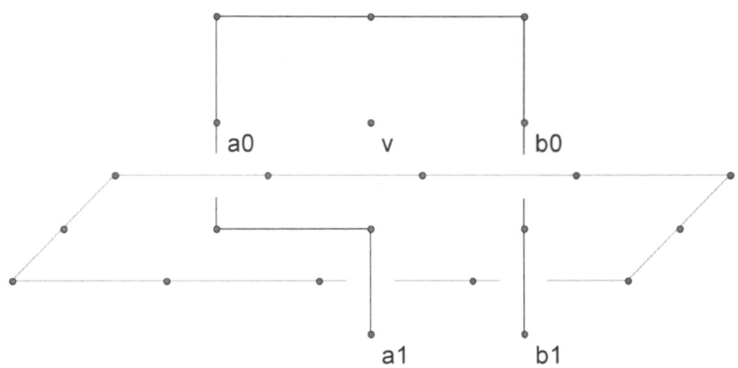

Figure 2.27

To form Q_1 (connect a_1 to v) will require at least 6 steps total (4 x-steps and 2 z-steps). To form Q_2 (connect b_1 to v) will use at least 7 steps (4 x-steps, 1 y-step, and 2 z-steps). Thus this configuration, shown in Figure 2.28, takes 34 steps total, and is identical to the one shown in Figure 1.11 (b). Once the proof is complete we will have shown that it is a minimal length embedding of the Whitehead link in the cubic lattice. Note we can perform a square move around vertex c_2 (see Figure 2.29) and have another minimal diagram where our two strands only pass through C_1 using vertices c_1 and c_3. (This is not the only square move we can perform.)

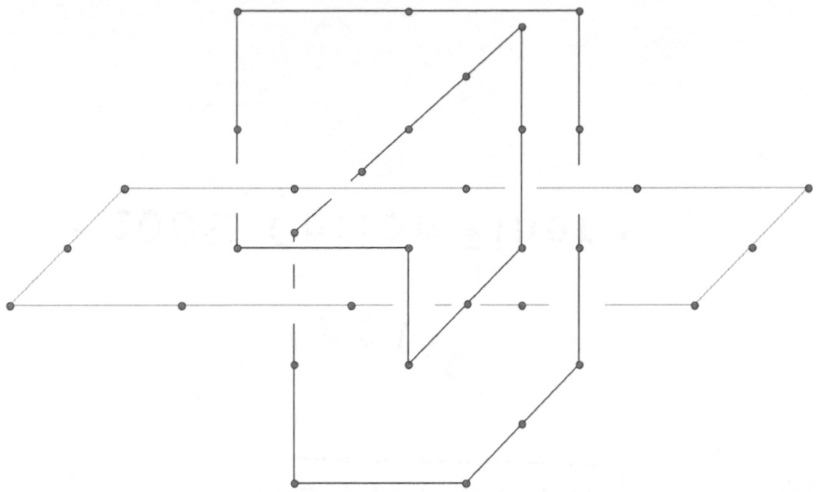

Figure 2.28

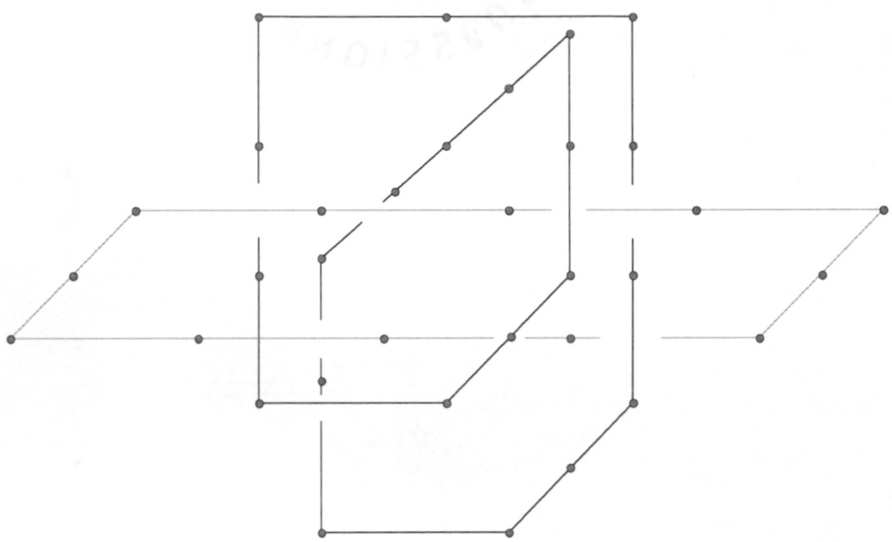

Figure 2.29

Case 2.1.2: Consider C_1 and the strands to be as in Figure 2.26. Given P_1 has length 6 (with respect to symmetry), there are two possibilities that are not immediately reducible, shown in Figure 2.30.

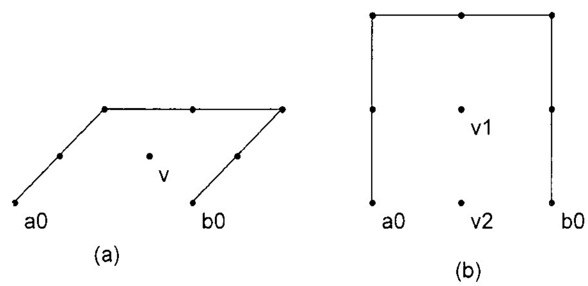

Figure 2.30

Due to the position of C_1, if P_1 is as shown in Figure 2.30 (a), P_2 cannot use the vertex v. Thus P_1 must be as shown in Figure 2.30 (b). Also, if P_2 passes only through v_2, then P_1 is reducible to the P_1 in Case 2.1.1. Now Q_1 (connecting a_1 to v) requires 5 steps (4 x-steps and 3 z-steps) and Q_2 (connecting b_1 to v) requires 8 steps (4 x-steps, 1 y-step, and 3 z-steps), so this diagram requires 38 steps total, which is not minimal. We leave the details to the reader.

Case 2.1.3: The path P_1 could have length 8. Since $P_1 \cup \overline{a_0 b_0}$ is a component of length 10, and $\overline{a_0 b_0}$ is 2 y-steps, we have the following cases to consider:

Sub-case	x-steps	y-steps	z-steps
1	0	2	6
2	0	4	4
3	0	6	2
4	2	2	4
5	2	4	2
6	2	6	0
7	4	2	2
8	4	4	0

Table 2.2

It can be shown that sub-cases 3 and 6 are immediately reducible, so we will not consider these cases.

Case 2.1.3 (1): Given P_1 containing 0 x-steps, 2 y-steps, and 6 z-steps, there is only one configuration that is not immediately reducible, shown in Figure 2.31.

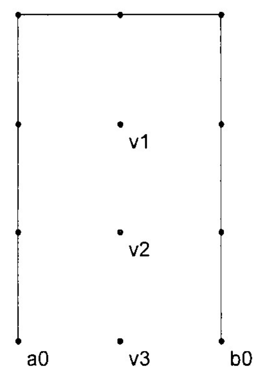

Figure 2.31

This P_1 is reducible unless P_2 contains v_1. Similar to Case 2.1.1, Q_1 will now require 8 steps and Q_2 will now require 9 steps, so this configuration has 42 steps, which is not minimal.

Case 2.1.3 (2): Given P_1 containing 0 x-steps, 4 y-steps, and 4 z-steps, there are two possibilities (with respect to symmetry) that are not immediately reducible and bound a vertex. These two are shown in Figure 2.32.

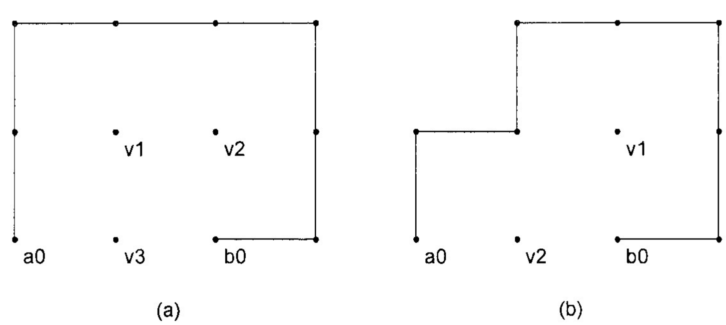

Figure 2.32

Consider P_1 as shown in Figure 2.32 (a). P_2 must contain v_2, or else P_1 is reducible to the P_1 in Case 2.1.1 or Case 2.1.2. This makes it very similar to the P_1

shown in Figure 2.32 (b). In each case, Q_1 requires 6 steps and Q_2 requires 7 steps, so this configuration has 40 steps total, which is not minimal.

Case 2.1.3 (4): Given P_1 with 2 x-steps, 2 y-steps, and 4 z-steps, then up to symmetry, P_1 must lie on the "ribbon" shown in Figure 2.33.

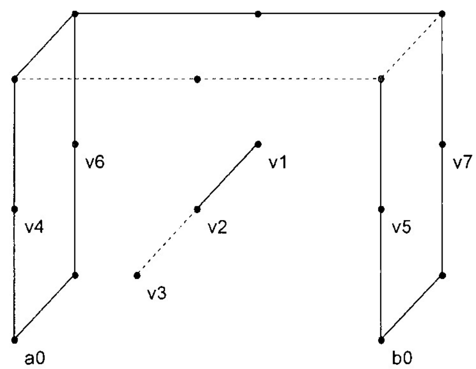

Figure 2.33

Regardless of the actual shape of P_1, the path P_2 must contain the edge $\overline{v_1v_2}$ and with respect to symmetry, we have the following five cases: 1) $\overline{v_2v_3} \in P_2$, 2) $\overline{v_2v_4} \in P_2$, 3) $\overline{v_2v_5} \in P_2$, 4) $\overline{v_1v_6} \in P_2$, or 5) $\overline{v_1v_7} \in P_2$. Cases 4 and 5 are similar to cases 2 and 3, so we will leave these to the reader.

Case 2.1.3 (4.1): $\overline{v_2v_3} \in P_2$, then P_1 is reducible to a path with no x-steps, as discussed in Case 2.1.2.

Case 2.1.3 (4.2): $\overline{v_2v_4} \in P_2$, then the path from a_1 to v_4 needs at least 8 steps (0 x-steps, 5 y-steps, and 3 z-steps or 4 x-steps, 1 y-step, and 3 z-steps) and the path from b_1 to v_1 needs at least 7 steps (3 x-steps, 1 y-step, and 3 z-steps), so the configuration contains at least 42 steps, which is not minimal.

Case 2.1.3 (4.3): $\overline{v_2 v_5} \in P_2$, then the path from a_1 to v_1 needs at least 4 steps (1 x-step and 3 z-steps) and the path from b_1 to v_5 needs at least 7 steps (4 x-steps and 3 z-steps), so the configuration requires 38 steps total, which is not minimal.

Case 2.2: Consider C_1 to be as shown in Figure 2.25 (b). To achieve a linking number of zero, either two strands pass through C_1 or four strands pass through C_1. First, we'll consider four strands passing through C_1, as shown in Figure 2.34.

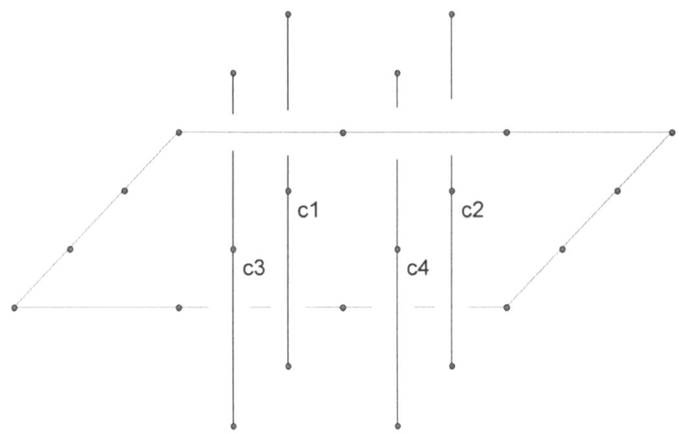

Figure 2.34

Assume a minimal configuration exists with four strands passing through C_1. In order to form a simple closed component C_2, four connecting paths must be added. To achieve an overall length less than or equal to 34, one of these paths must have length 3 or less (length(P) $\leq \frac{34-12-8}{4} = 3.5$). However this implies that C_2 is reducible and does not need to be considered further. Next, we consider two strands passing through C_1.

In order for C_1 to not be reducible to length 10, these two stands must pass through c_1 and c_4 or c_2 and c_3. Without loss of generality, assume C_2 passes through c_2 and c_3, as shown in Figure 2.35.

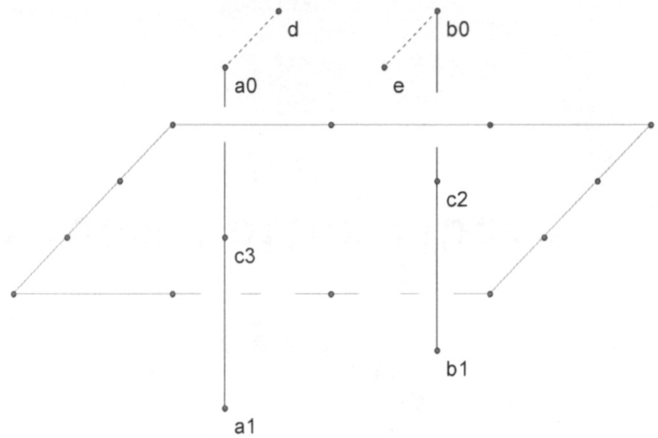

Figure 2.35

P_1 must have an even number of steps since two steps separate a_0 and b_0. Since a_0 and b_0 do not lie on a line parallel to the x-, y-, or z-axis, we will use vertices d and e (see Figure 2.35) to fulfill Definition 1.17. Since a_0 and d (a_0 and e) are separated by one x-step (y-step), any path S that bounds a vertex contains at least seven steps. Since $P_1 = S \cup \overline{a_0 d}$ $(\overline{a_0 e})$ and length of $P_1 \leq \dfrac{34-12-4}{2} = 9$, P_1 must have length 8. Due to symmetry, we need only consider P_1 as shown in Figure 2.36.

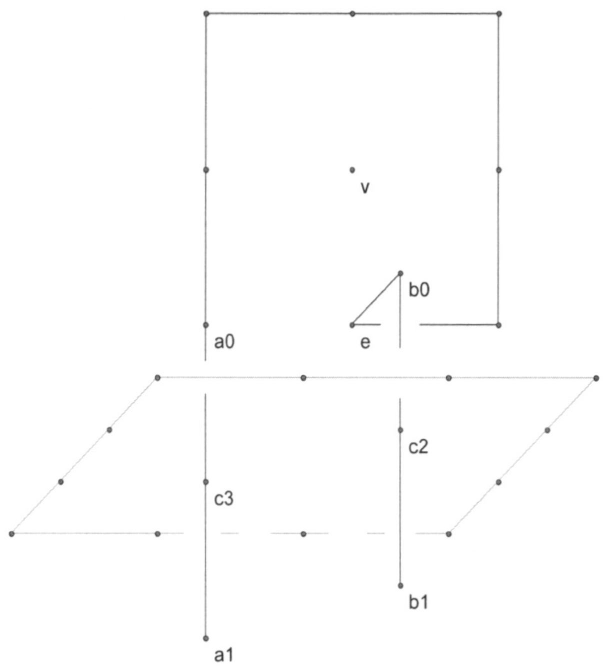

Figure 2.36

Q_1 must have length 8 (4 x-steps, 1 y-step, and 3 z-steps), Q_2 must have length 8 (5 x-steps, 3 z-steps), and so this configuration requires 40 steps total, which is not minimal.

Case 2.3: The remaining two cases of Figure 2.25 are very similar, so we will only discuss C_1 as shown in Figure 2.25 (d). As in Case 2.2, P_1 must have length 8. The only way this case could be different from Case 2.2 is if C_2 contains vertices d or e as shown in Figure 2.37.

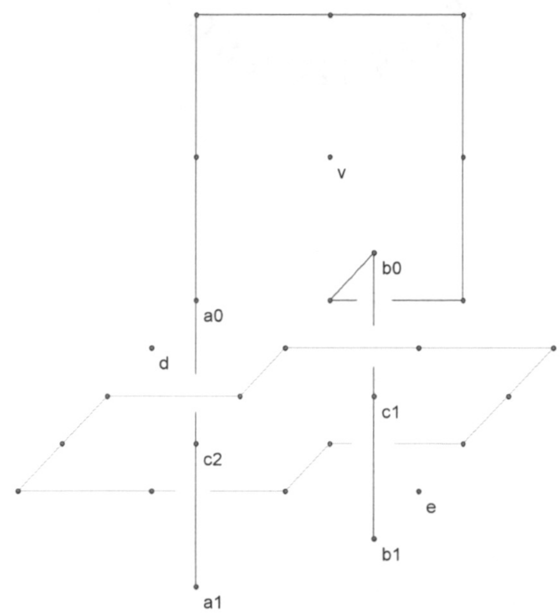

Figure 2.37

To connect a_1 to d will require 4 steps (2 x-steps, 1 y-step, and 1 z-step), to connect d to v will require at least 6 steps (2 x-steps, 2 y-steps, and 2 z-steps) based on the differences in their coordinates. Thus Q_1 contains 10 steps for a total of 34 steps so far in the construction. Adding Q_2 will yield a length greater than 34.

Case 2.5: Now we will consider the cases where neither component is planar. The only way two strands can pass through component C_1 is if the projection of C_1 along the x-, y-, or z-axis will allow two strands to pass through. Then without loss of generality, the projection of C_1 onto the x-y plane must look like the C_1 of Case 1. Thus C_1 lies on the "ribbon" as shown in Figure 2.38. Regardless of the actual shape of C_1, the two edges $\overline{a_0a_1}$ and $\overline{b_0b_1}$ will be contained in C_2.

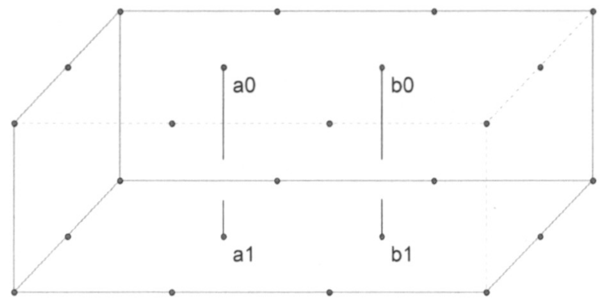

Figure 2.38

Depending upon the actual shape of C_1, C_2 must contain the two lattice paths from a_0 to a_2 and b_0 to b_2, that are arranged as one of the following 4 cases (up to symmetry), shown in Figure 2.39. We will call the path from a_0 to a_2 R_1 and the path from b_0 to b_2 R_2.

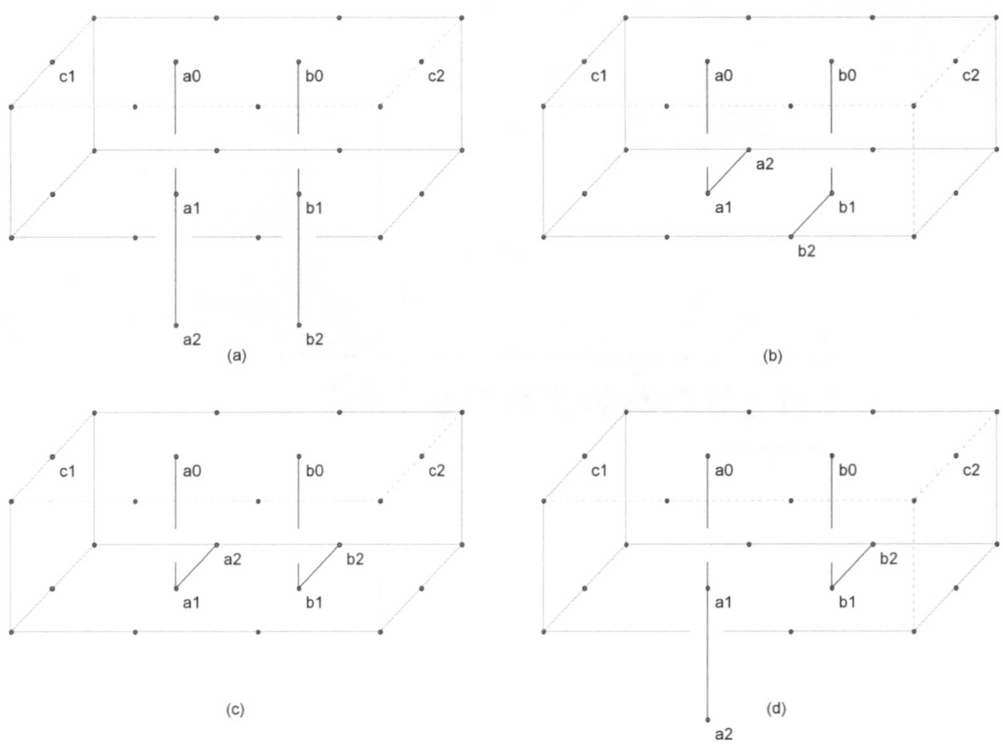

Figure 2.39

Since a_0 and b_0 are one y-step apart, we know P_1 has a minimum of 7 steps. Also, we know that length(P_1) $\leq \frac{34-12-4}{2} = 9$. If both $c_1, c_2 \in C_1$, then P_1 must have at least 9 steps and look like the path shown in Figure 2.40c (up to symmetry). A 9 step P_1 leaves at most 9 steps for $P_2 = Q_1 \cup Q_2$ in a lattice embedding with ≤ 34 steps. In such a case, Q_1 or Q_2 must have length ≤ 4 steps which is not possible. The details are left to the reader (they are similar to the analysis in Case 1.2).

Now we will assume that P_1 has 7 steps as shown in Figure 2.40 (a) and (b). Note that in their cases either $c_1 \notin C_1$ or $c_2 \notin C_1$.

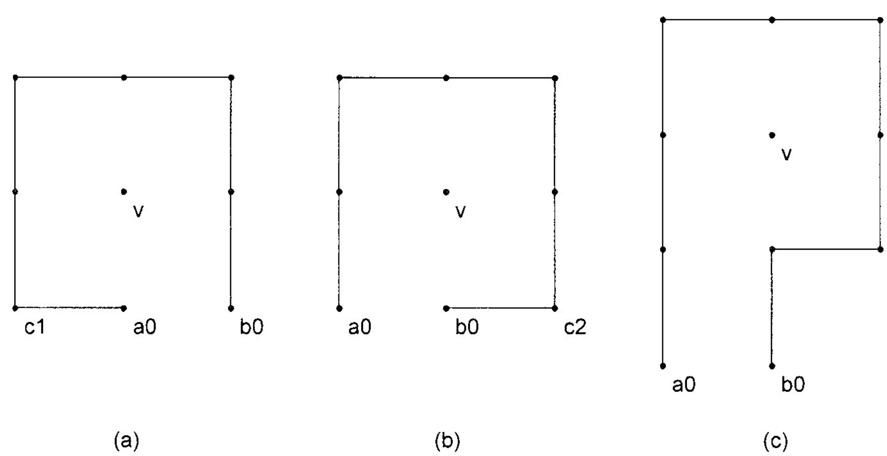

Figure 2.40

Case 2.5.1: Consider C_1, R_1, and R_2 to be as shown in Figure 2.39 (a). Up to symmetry, $c_1 \notin C_1$ and $c_2 \notin C_1$ are the same case. So without loss of generality, we will discuss the case $c_1 \notin C_1$, this gives us a configuration as shown in Figure 2.41.

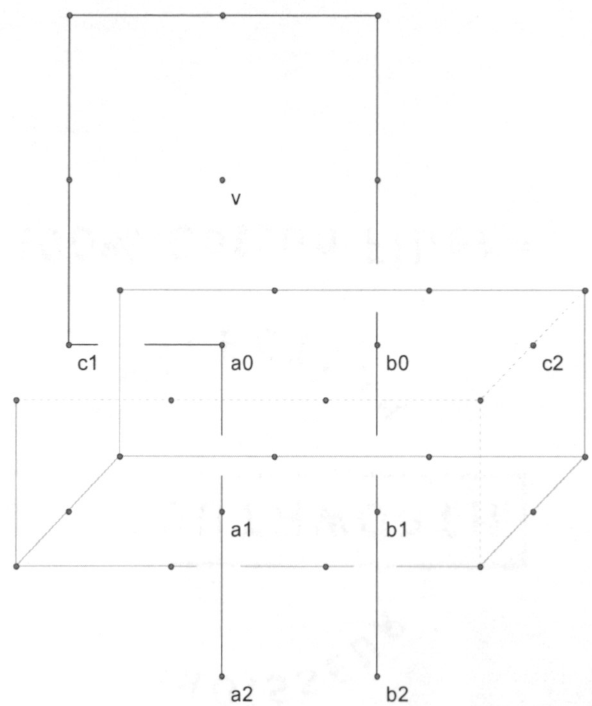

Figure 2.41

To form Q_1 (the path that connects a2 to v) will have a minimum of 7 steps (4 x-steps and 3 y-steps). Q_2 (the path from b2 to v) will require 8 steps (4 x-steps, 1 y-step and 3 z-steps). Thus this configuration will have length 38, which is not minimal.

Case 2.5.2: Consider C_1, R_1, and R_2 to be as shown in Figure 2.39 (b). Consider $c_1 \notin C_1$, then P_1 must be as shown in Figure 2.40 (a) (the case of P_1 as shown in Figure 2.40 (b) is the same up to symmetry). Then this gives us a configuration as shown in Figure 2.42.

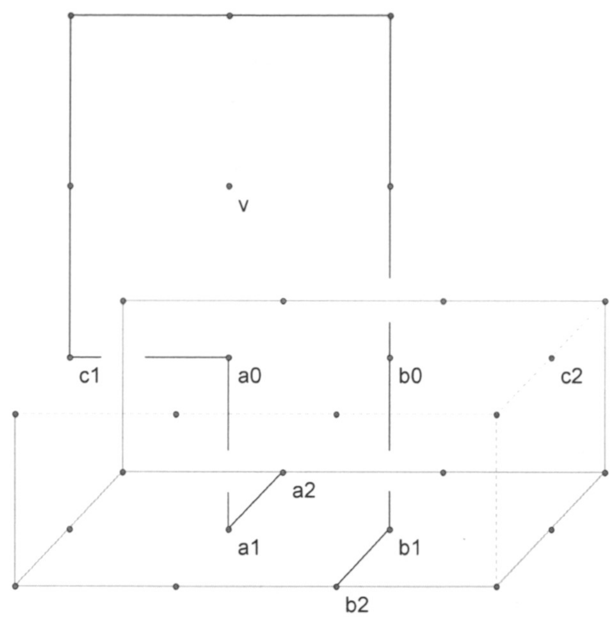

Figure 2.42

Q_1 will require 5 steps (3 x-steps and 2 z-steps). Q_2 will require 6 steps (3 x-steps, 1 y-step and 2 z-steps). Thus our configuration has 34 steps total, so our proof will show that this will result in a minimal diagram. Figure 2.43 gives an example of such a C_1 and C_2.

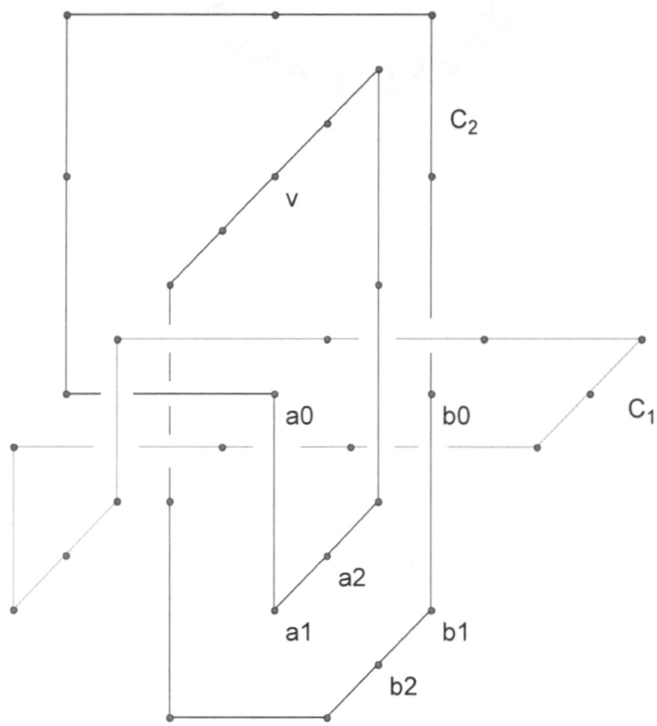

Figure 2.43

Case 2.5.3: Consider C_1, R_1, and R_2 to be as shown in Figure 2.39 (c). Due to symmetry, we will discuss only the case $c_1 \notin C_1$. Then P_1 must be as shown in Figure 2.44.

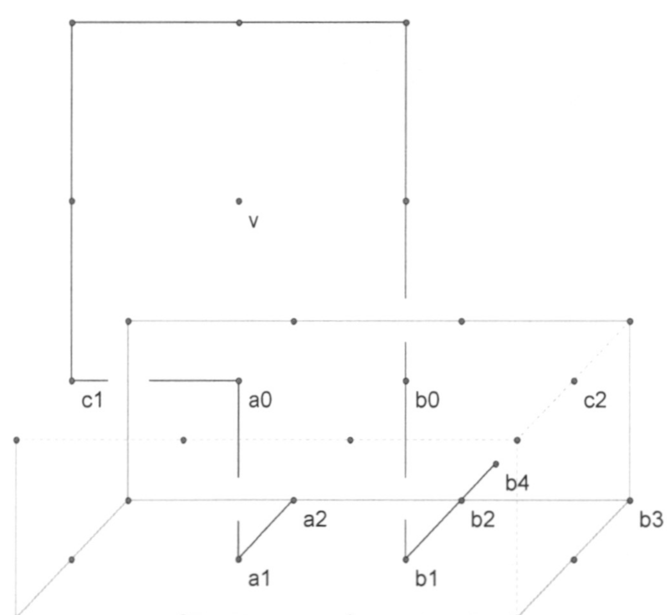

Figure 2.44

Q_1 will require 5 steps (3 x-steps and 2 z-steps). The shape of Q_2 could depend upon the actual shape of C_1. If $\overline{b_2b_3} \in Q_2$, then Q_2 will require 10 steps (3 x-steps, 5 y-steps and 2 z-steps). If $\overline{b_2b_4} \in Q_2$, then Q_2 will require 10 steps (5 x-steps, 3 y-steps and 2 z-steps). So no matter the shape of Q_1, this gives a representation with total length 38, which is not minimal.

Case 2.5.4: Consider C_1, R_1, and R_2 to be as shown in Figure 2.39 (d). Since R_1 and R_2 are not symmetric, we have to consider the cases of Figure 2.40 (a) and (b) separately.

Case 2.5.4 (1): Given $c_1 \notin C_1$, P_1 must be as shown in Figure 2.45.

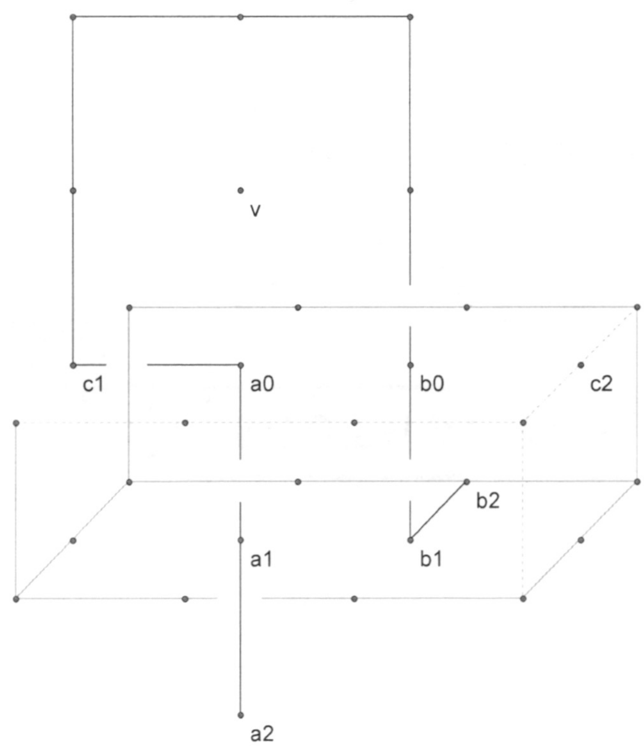

Figure 2.45

Q_1 will require 7 steps (4 x-steps and 3 z-steps). Q_2 will require 6 steps (3 x-steps, 1 y-step and 2 z-steps). Thus our configuration has 36 steps total, so it is not minimal.

Case 2.5.4 (2): Given $c_2 \notin C_1$, P_1 could be as shown in Figure 2.46.

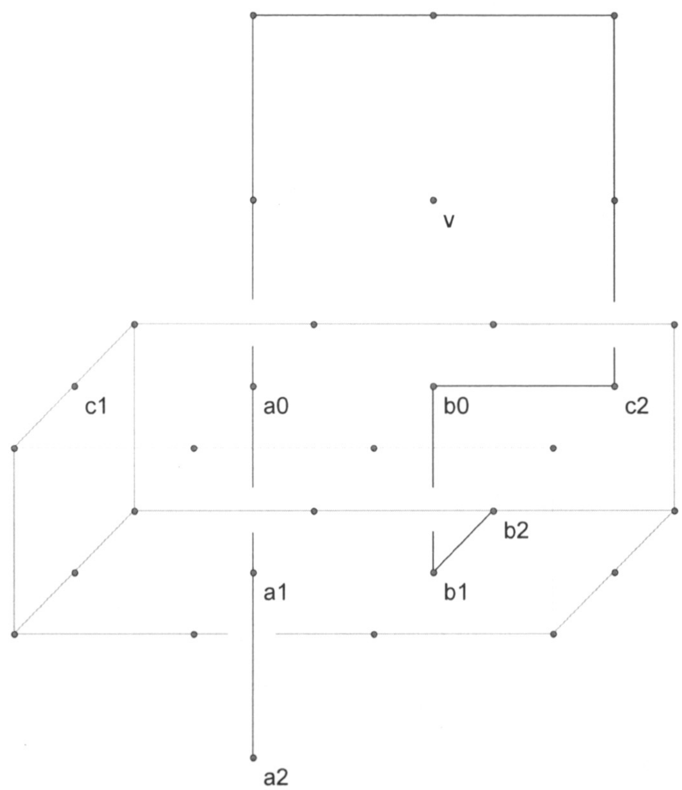

Figure 2.46

Q_1 will require 8 steps (4 x-steps, 1 y-step and 3 z-steps). Q_2 will require 5 steps (3 x-steps, and 2 z-steps). Thus our configuration has 36 steps total, so it is not minimal.

Case 3: C_1 has length 14

The main difficulty of Case 3 (and the following Case 4) is that there are many different components of 14 steps (and 16 steps) when compared to the number of 10 and 12 step components. We classify these components by their projection in a lattice plane. However, even then there are many different cases of planar projections, see for example the 28 cases in Figures 2.58-2.60, 2.64, 2.67. There is no general method to eliminate all 28 cases. Each planar configuration needs to be considered separately. In this thesis we will give the details of some of these cases to show the types of arguments that are needed. We choose not to write up the details of all cases (this would fill a book!), but to leave it the reader to fill in the necessary details if the reader is so inclined.

For this case (and the next case), it suffices to show the Whitehead Link cannot be constructed with 32 steps or fewer if one component has length 14. While this cuts down the number of cases to consider, it has the disadvantage that we cannot rule out the existence of lattice embeddings of the Whitehead link with 34 steps when one component has length 14 or 16. Note that if the Whitehead link can be constructed with 32 steps and one component has length 14 then the other component has length 18.

Definition 2.1: Let C be a simple closed lattice curve that is one component of a lattice link L of several components. Let P be a projection along the x-, y-, or z-axis. By varying the projection ever so slightly, we can assume that P is a regular knot projection (see Definition 1.4). Let c be a crossing in P. The crossing c is a <u>reducible crossing</u> if one of the following holds:

1) There exists a small $\varepsilon > 0$ and an ε-perturbation C' of C (moving C off lattice) such that the projection P changes to a projection P', where c is not a crossing in P'.

2) c can be eliminated by a Reidermeister I move on the projection of the whole link L (moving C off lattice) such that the projection P changes to a projection P', where c is not a crossing in P'.

We say c is irreducible if c is not reducible.

Lemma 2.1: The Whitehead Link cannot contain a component C_1 such that all the following conditions hold:

1) Length $(C_1) \leq 18$
2) A projection of C_1 contains an irreducible crossing
3) Length (Whitehead Link) ≤ 34.

Proof of Lemma 2.1: (by contradiction)

Without loss of generality, assume C_1 contains an irreducible crossing A when projected into the y-z plane. Let P be the projection. If one travels along P, starting at A, until one returns to A, then one travels along a loop in the x-y plane. Note that this loop cannot have only 4 y-z steps. If it does, the loop cannot contribute to any crossing of C_1 with C_2 regardless of how many x-steps are inserted, thus the crossing A would be reducible by a Reidermeister I move (see Figure 1.2). This loop must have either 6 or 8 y-z steps. Thus up to symmetry, P could look as one of the following, shown in Figure 2.47 (recall that C_1 has at most 18 steps).

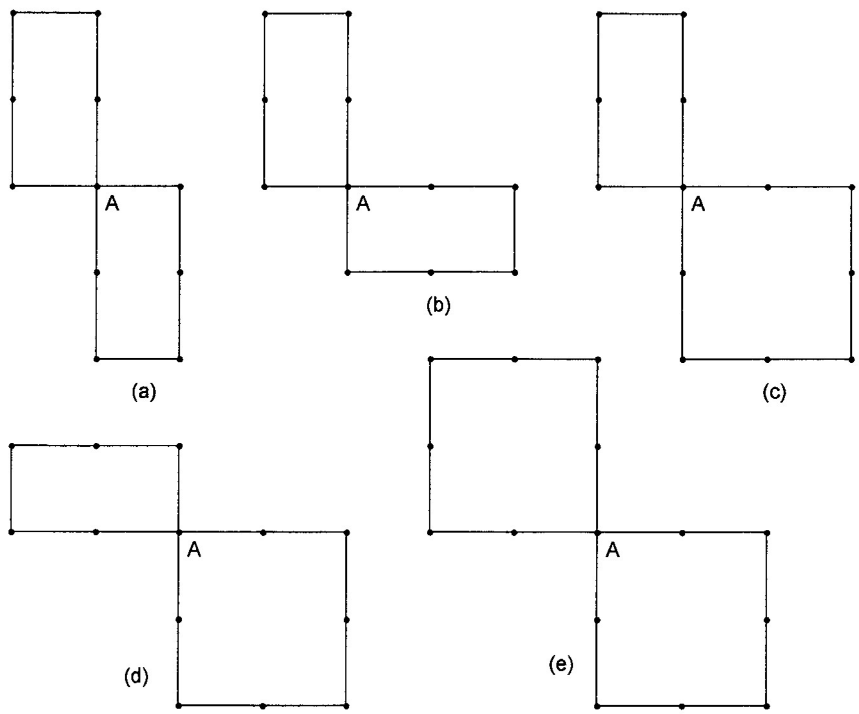

Figure 2.47

In case (e), at least 2 x-steps are required in order for the projection to be capable of linking with C_2. In cases (a) - (d), at least 4 x-steps are needed; otherwise the 6-step loop in the projection cannot contribute to the linking with C_2, forcing the crossing A to be reducible. Thus C_1 must have at least 16 steps in cases (a) and (b) and 18 steps in cases (c) - (e). Any component in Figure 2.47 can be used to build the Whitehead link. However, the question is raised of whether the length of C_2 can be small enough to have an overall length of at most 34 steps.

Consider C_1 as shown in Figure 2.47 (c). As before, there must be at least two strands of C_2 that pass through C_1 creating a linking number of zero. Moreover the projection of C_2 must intersect both loops of C_1 since the crossing in the projection of C_1 is irreducible. Two possible projections of C_2 are shown in Figure 2.48.

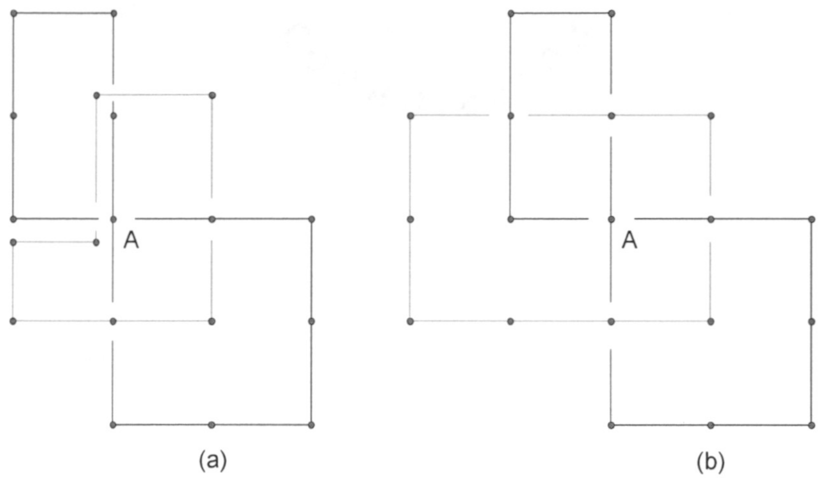

Figure 2.48

When we add z-steps to the C_2 of Figure 2.48 (a), we realize that as an alternating projection, it is impossible; an 18-step projection cannot form the Whitehead link with the 6-step loop in the projection of C_1. (The same is true if one performs a square move in the upper left corner in the projection of C_2; the details are left to the reader.) Thus the projection must contain at least six crossings. (See also Figure 2.52.) One can check that we cannot create such a six crossing configuration with the available 18 steps.

Figure 2.48 (b) shows an alternating projection of the Whitehead link with C_1 as shown in Figure 2.47 (c). This C_2 appears possible until we add x-steps to create an alternating diagram. Without loss of generality, we give the vertical line segment of C_1 an x-coordinate of zero. Figure 2.49 shows the projection of the Whitehead link on the y-z plane with each line segment labeled with the minimal x-coordinate necessary to create an alternating diagram.

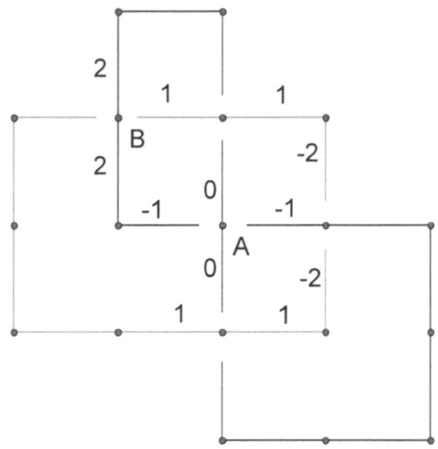

Figure 2.49

Note at Crossing B of Figure 2.49, C_1 must have extra x-steps to make this an alternating diagram (and any 5 crossing projection of the Whitehead link must be alternating), thus C_1 contains over 18 steps, a contradiction. There are, of course, many other projections of C_2 as the two discussed here, but all of these make C_2 even longer.

Similarly, one can argue the cases (a) - (d) of Figure 2.47. In all cases, the Whitehead link formed with such a C_1 will require over 34 steps total. (The details are left to the reader.)

Consider C_1 as shown in 2.48 (e). The shortest possible projection of C_2 must contain at least 8 steps, with two strands passing through C_1, as shown in Figure 2.50.

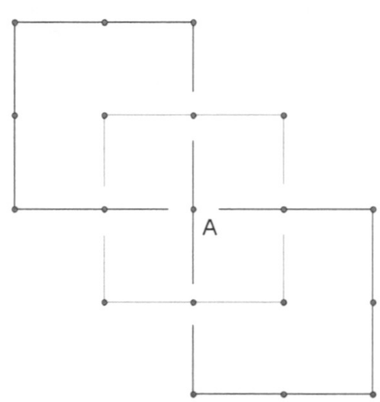

Figure 2.50

Once again, we must add x-steps to form an alternating diagram. Without loss of generality, we give the vertical line segment of C_1 an x-coordinate of zero. Figure 2.51 shows the projection of the Whitehead link on the y-z plane with each line segment labeled with the minimal x-coordinate necessary to create an alternating diagram.

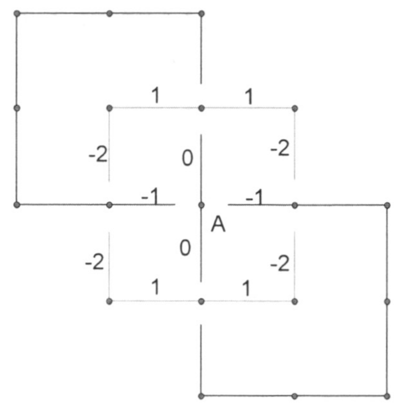

Figure 2.51

Thus C_2 has length 20, and the length of the Whitehead link is now 38, a contradiction. Other projections of C_2 are possible, but they lead to configurations of the Whitehead link with more than 34 steps (details are left to the reader). Hence it is impossible to form the Whitehead link such that:

1) Length $(C_1) \leq 18$
2) A projection of C_1 contains an irreducible crossing
3) Length (Whitehead Link) ≤ 34. □

From Lemma 2.1, we know that if a 32 step (or fewer) configuration of the Whitehead link exists with length$(C_1) = 14$, neither component can have an irreducible crossing. Figure 2.52 shows a deformation of the Whitehead link to a diagram where neither component has an irreducible crossing. Note that while the linking number between the two components is still zero, there are now 6 or more crossings in any such diagram.

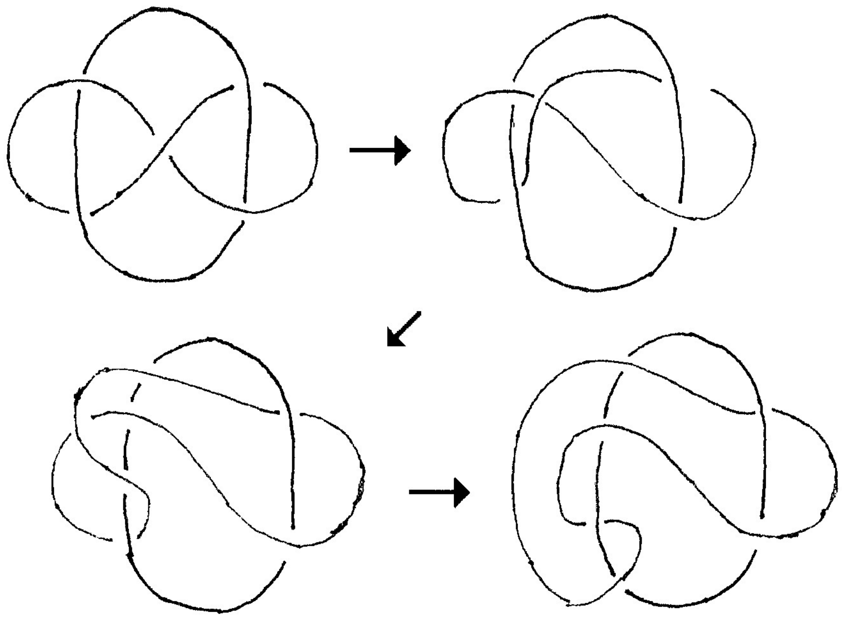

Figure 2.52

The following lemma is technical; however its importance will become clear as we discuss this case (see Case 3.1).

Lemma 2.2: Let A and B be two lattice planes such that distance(A,B) = 2. No simple closed lattice curve C_2 of length 18 that intersects A to B by 4 paths can be part of a Whitehead lattice link L with length(L) < 34.

Proof of Lemma 2.2:

Let the projection of the planes be as shown in Figure 2.53.

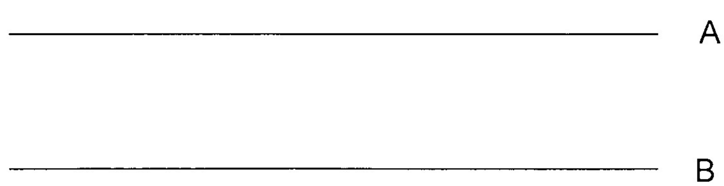

Figure 2.53

Case 1: Assume that the z-extend of C_2 is 3 units. This implies that C_2 must contain at least 10 z-steps. (C_2 contains 4 paths of z-steps. Two paths have length 3 and two paths have length 2.) Note that if C_2 has length 18 and contains more than 10 z-steps, then C_2 must be reducible. The details are left to the reader.

Thus C_2 contains exactly 10 z-steps. The only way that C_2 is not immediately reducible is if the following occurs when we walk along C_2: 3 z-steps, 2 x- or y-steps, 3 z-steps, 2 x- or y-steps, 2 z-steps, 2 x- or y-steps, 2 z-steps, and 2 x- or y-steps. If any of the 2 x- or y-step pairs consist of one x- and one y-step, then C_2 can't bound two vertices (details left to the reader). Thus C_2 must look like Figure 2.54. Note that no square move on C_2 is possible (all such moves make C_2 reducible).

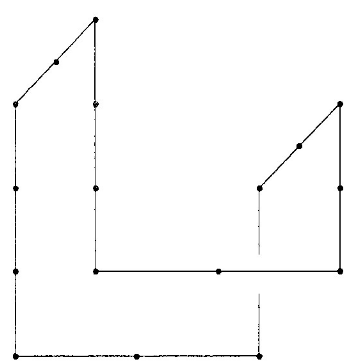

Figure 2.54

We now show that C_1 must have a length > 14.

After an ε-deformation of C_2, we can project C_2 into the x-y plane as shown in Figure 2.55.

There have to be at least 6 intersections (crossings) with C_2, which means the projection of C_1 passes through x at least 3 times. In the projection into the x-y plane it takes at least 8 steps for a non-reducible C_1 to return to x, so length(C_1) > 14.

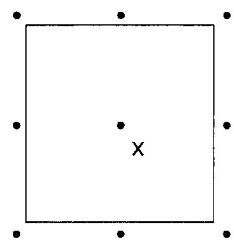

Figure 2.55

Case 2: Assume that the z-extend of C_2 is 2 units.

Case 2.1: Assume that the x-extend and y-extend of C_2 is only 2 units. Thus C_2 is contained in a cube. No curve of length 18 in this cube can bound 2 vertices. (Details are left to the reader.)

Case 2.2: Assume that the x-extend or the y-extend of C_2 is 3 steps. (It can only be one of the two extends.) Without loss of generality, let it be the y-extend.

Similar to the discussion in Case 1, the only way that C_2 can bound two vertices is as shown in Figure 2.56.

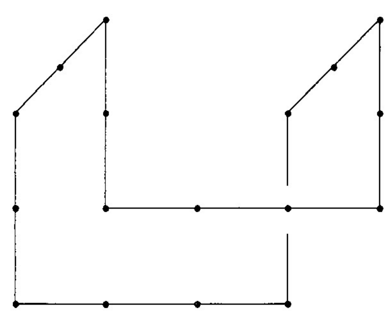

Figure 2.56

We can project this into the x-z plane as shown in Figure 2.57.

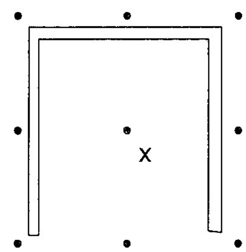

Figure 2.57

The only possibility to generate the needed ≥ 6 intersections is for C_1 to intersect x at least two times in this projection. This implies length(C_1) > 14, for the same reason as in Case 1. □

Without loss of generality, assume that the number of y-steps of C_1 is less than or equal to the number of x-steps of C_1, which is less than or equal to the number of z-steps of C_1. Then the possible combinations of C_1 are listed below.

x-steps	y-steps	z-steps
2	0	12
4	0	10
6	0	8
2	2	10
4	2	8
6	2	6
4	4	6

Table 2.3

Case 3.1: C_1 has 14 steps and is planar

A projection of a planar C_1 using a projection direction in the plane of C_1 would be a line segment. Lemma 2.1 tells us C_2 cannot contain an irreducible crossing if length(Whitehead Link) ≤ 34. Thus this projection must contain at least 6 crossings (see Figure 2.52). To form 6 crossings, C_2 would contain 3 paths each containing 2 steps, connecting Plane A to Plane B, see Figure 2.58.

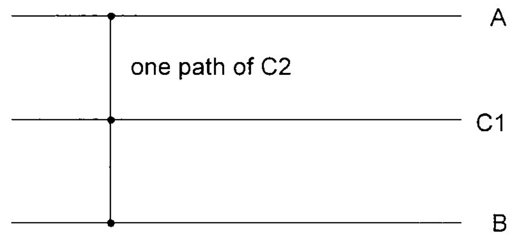

Figure 2.58

In order to make C_2 be a closed component, C_2 must contain an even number of these paths, thus C_2 must contain at least 4 such paths. Lemma 2.2 tells us we cannot connect 4 such paths so that length(C_2) ≤ 18, therefore we cannot form a representation of the Whitehead link with 32 steps where C_1 has length 14 and is planar.

Case 3.2: C_1 has 2 x-steps, 2 y-steps, and 10 z-steps

Figure 2.59 shows the possible projections up to symmetry onto the x-y plane where C_1 has 2 x-steps, 2 y-steps, and 10 z-steps. Note that it is impossible for any such component to link with another component, thus they cannot be part of any link.

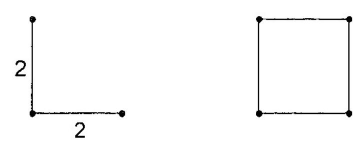

Figure 2.59

Case 3.3: C_1 has 4 x-steps, 2 y-steps, and 8 z-steps

Figure 2.60 shows the possible projections onto the x-y plane (with respect to symmetry) of a C_1 with 4 x-steps, 2 y-steps, and 8 z-steps. In this thesis, we will discuss the projection of C_1 as shown in Figure 2.60 (a) and (e). To rule out cases of the projection of C_1 as shown in Figure 2.60 (b), (c), and (d) is not trivial, and no more difficult than the case of Figure 2.60 (a); the details of cases (b) – (d) are left to the reader.

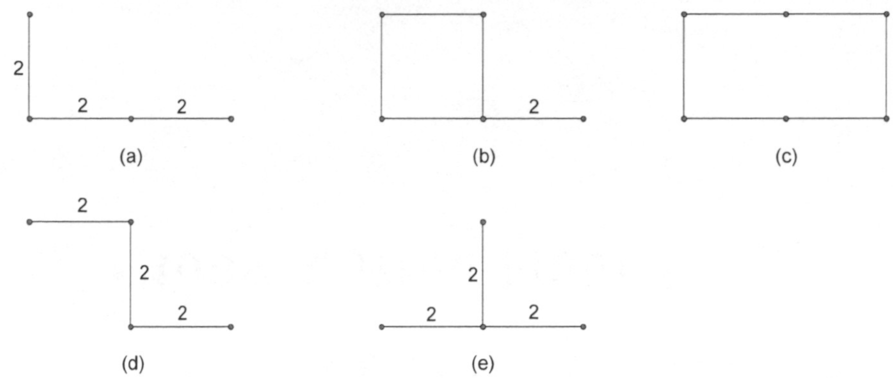

Figure 2.60

Consider the projection of C_1 as shown in Figure 2.60 (a). By Lemma 2.1, C_2 cannot contain an irreducible crossing, thus C_2 contains at least three paths passing through (or above/below) C_1 to generate 6 crossings. Each path must begin at the vertex v and could be continued in one of three ways: as shown in Figure 2.61 (a) – (c). Thus these three paths will use at least 6 steps of C_2.

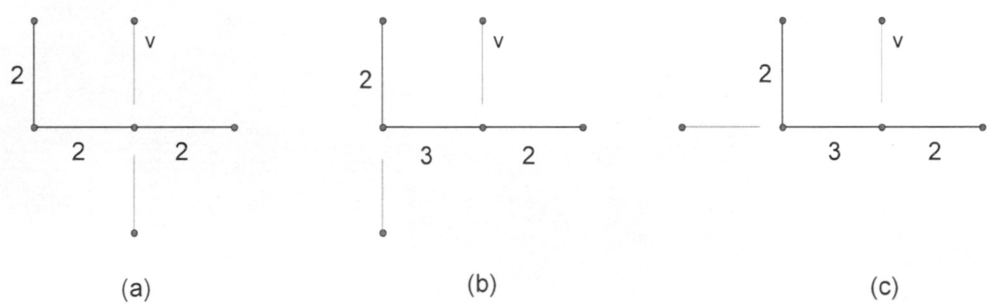

Figure 2.61

We know at least two of these paths pass through C_1. Let these paths be named path a_1 and path a_2. If a_1 and a_2 have the same orientation, then we will need two more paths passing through C_1 oriented opposite of a_1 and a_2 to generate a linking number of zero. However, since C_1 has only 8 z-steps, we have at most 3 strands that can pass through it. Thus a_1 and a_2 cannot have the same orientation; they must have opposite orientations. For that same reason, our third path (path b) cannot pass through C_1. Without loss of generality, we will say b crosses above C_1.

Without loss of generality, let a_1 have the same orientation as b. Then it will take at least 5 steps in the x-y plane to connect a_1 and b in a non-reducible connection, see Figure 2.62. (Note that this length is needed because the connection needs to go "around" C_1.)

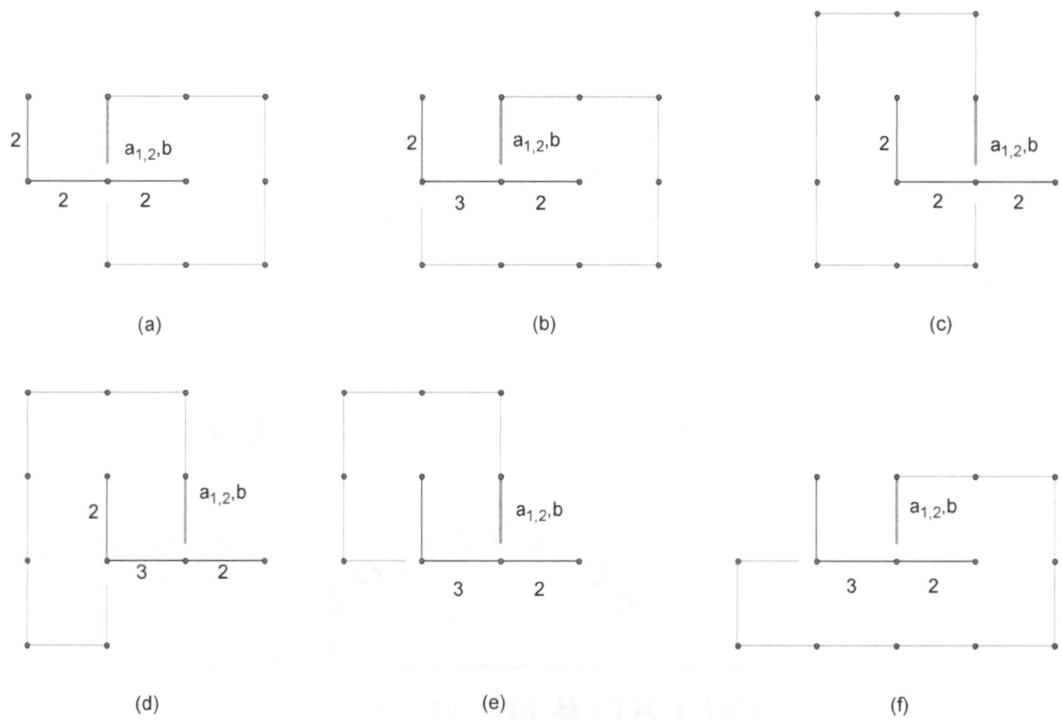

Figure 2.62

The paths a_1 and a_2 must be separated by 2 z-steps otherwise C_1 is reducible (C_1 has 8 z-steps and there are only 6 needed if a_1 and a_2 are separated by 1 z-step). Also, path b must be 2 z-steps above a_1 or a_2 (thus 4 z-steps above the other). The three paths have a total length of at least 6 steps and the connection between a_1 and b also has a length of at least 5 steps. In order for length(C_2) ≤ 18, we have at most 7 steps remaining.

These remaining 7 steps are not enough to connect these three paths in such a way that is not immediately retractable (details are left to the reader).

The case of Figure 2.60 (e) is easier than the case just discussed. Consider the projection of C_1 as shown in Figure 2.60 (e). Then up to symmetry (and without an irreducible crossing), C_1 could be as shown in Figure 2.63.

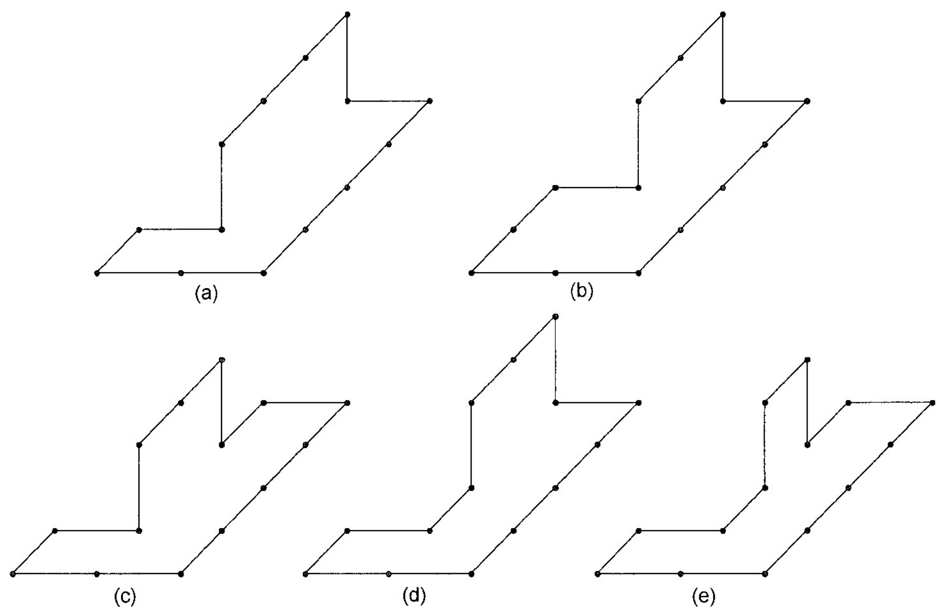

Figure 2.63

All configurations are reducible except Figure 2.63 (b), however C_1 as shown in Figure 2.63 (b) will only allow one strand to pass through, and thus this cannot be part of the Whitehead link.

The cases of Figure 2.60 (b) - (d) are entirely left to the reader. None of them can be used to construct a Whitehead link that is short enough.

Case 3.4: C_1 has 6 x-steps, 2 y-steps, and 6 z-steps

Figure 2.64 shows the possible projections onto the x-y plane (with respect to symmetry) of a C_1 with 6 x-steps, 2 y-steps, and 6 z-steps.

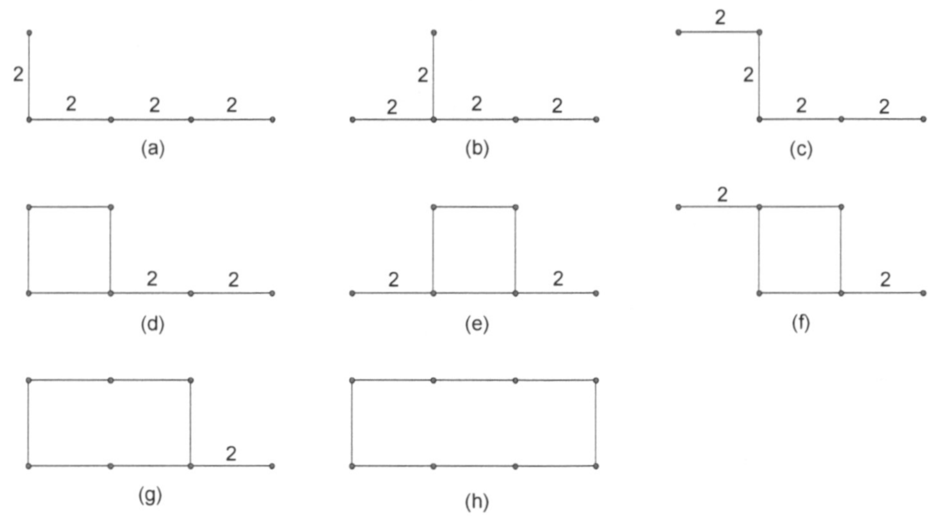

Figure 2.64

We first consider the projection of C_1 as shown in Figure 2.64 (a). As in Case 3.4, we know C_2 contains at least 3 paths (we will once again name them a_1, a_2, and b); each path has eight possibilities, as shown in Figure 2.65.

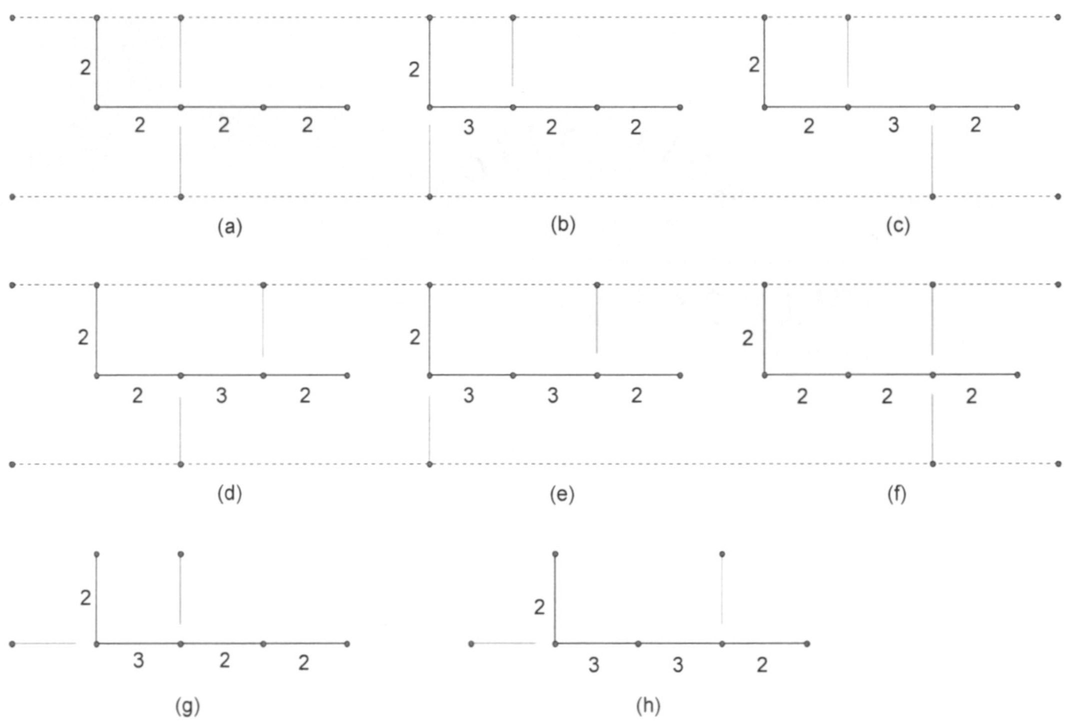

Figure 2.65

Note that in cases (a) - (f) of Figure 2.65, each path is intersecting two lattice planes A and B (represented by dashed lines in Figure 2.65) where distance(A,B) = 2. If each of our three paths projects to one of these six cases, then to close the curve C_2 there must exist a fourth path that also intersects planes A and B. Thus by Lemma 2.2, we know this cannot form the Whitehead link.

Assume there exists at least one path as shown in Figure 2.65 (g). In order for C_1 to be non-reducible, there must exist a path (a_1, a_2, or b) that contains a vertex that projects to v_1, as shown in Figure 2.66.

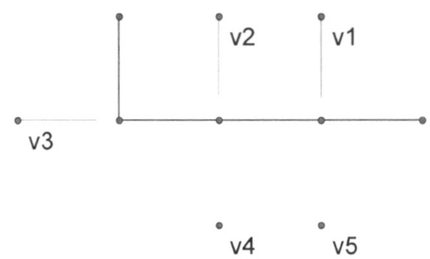

Figure 2.66

Case 1: Assume v_4 and v_5 (See Figure 2.66) are not used in the projection of C_2. Then length(a_1) + length(a_2) + length(b) $\geq$ 10. Then we have 8 steps remaining to form 3 connections if length(C_2) = 18. Thus there exists a path P such that length(P) $\leq \frac{8}{3}$.

Hence length(P) = 1 or 2, which will force C_2 to be reducible (details left to the reader).

Case 2: Assume v_4 is not used in the projection of C_2, but v_5 is used in the projection of C_2. Then length(a_1) + length(a_2) + length(b) $\geq$ 7. So that C_2 is not retractable, there must exist connections P_1-P_3 between the paths a_1, a_2, and b that go "around" C_1. The two shortest such paths are from v_2 to v_3 (a minimum projection of 3 steps) and from v_1 to v_5 (a minimum projection of 4 steps). If length(C_2) = 18, we now have at most 4 steps remaining for our third connection. These four steps are not enough to fill in the z-steps necessary for paths P_1, P_2, and to create a P_3, thus this projection cannot occur in a minimal diagram where C_1 has length 14. Details are left to the reader.

Case 3: Assume v_5 is not used in the projection of C_2, but v_4 is used in the projection of C_2. Then length(a_1) + length(a_2) + length(b) ≥ 7. So that C_2 is not retractable, there must exist connections P_1-P_3 between the paths a_1, a_2, and b that go "around" C_1. Also, we know from Lemma 2.1 that C_2 cannot have an irreducible intersection, thus a connection from one vertex to another cannot intersect any of our paths a_1, a_2, or b. The two shortest such paths that meet these criteria are from v_2 to v_3(a minimum projection of 3 steps) and from v_1 to v_4 (a minimum projection of 5 steps). If length(C_2) = 18, we now have at most 3 steps remaining for our third connection. These three steps are not enough to fill in the z-steps necessary for paths P_1, P_2, and to create a path P_3, thus this projection cannot occur in a minimal diagram where C_1 has length 14.

Case 4: Assume v_4 and v_5 are used in the projection of C_2. Then length(a_1) + length(a_2) + length(b) ≥ 7. So that C_2 is not retractable, there must exist connections P_1-P_3 between the paths a_1, a_2, and b that go "around" C_1. Also, we know from Lemma 2.1 that C_2 cannot have an irreducible intersection, thus a connection from one vertex to another cannot intersect any of our paths a_1, a_2, or b. The two shortest such paths that meet these criteria are from v_2 to v_3 (a minimum projection of 3 steps) and from v_1 to v_5 (a minimum projection of 4 steps). If length(C_2) = 18, we now have at most 4 steps remaining for our third connection. These four steps are not enough to fill in the z-steps necessary for paths P_1, P_2, and to create a path P_3, thus this projection cannot occur in a minimal diagram where C_1 has length 14.

Therefore it is impossible to create a representation of the Whitehead link with 32 steps with a projection of C_1 as shown in Figure 2.64 (a). The discussion of cases Figure 2.64 (b) - (h) is similar in difficulty and left entirely to the reader.

Case 3.5: C_1 has 4 x-steps, 4 y-steps, and 6 z-steps

Figure 2.67 shows the possible projections onto the x-y plane (with respect to symmetry) of a C_1 with 4 x-steps, 4 y-steps, and 6 z-steps.

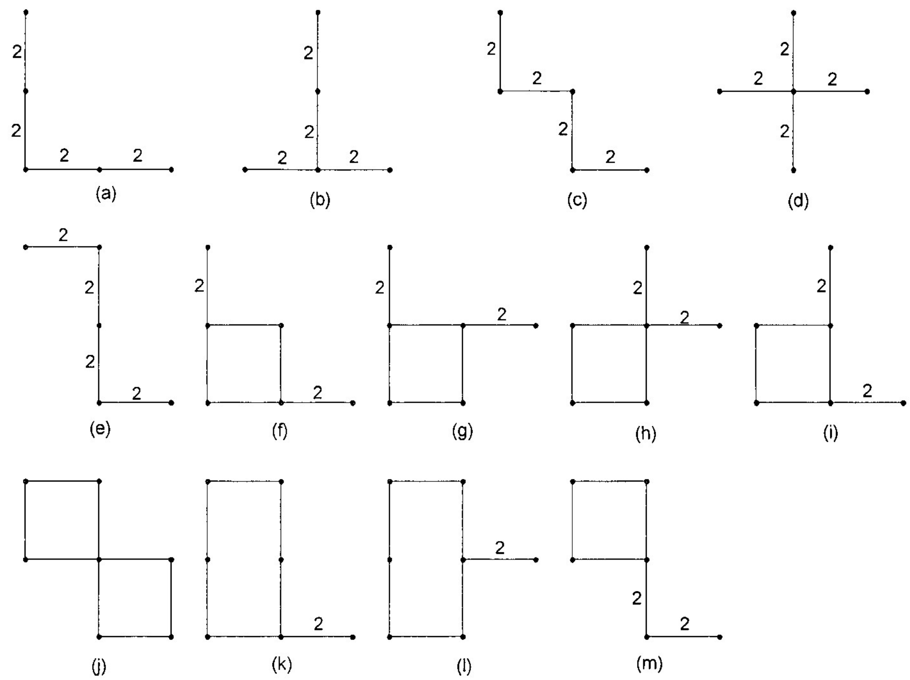

Figure 2.67

We first consider the projection of C_1 as shown in Figure 2.67 (a). As before, there are 3 paths crossing the projection of C_1 to generate the required 6 crossings. Each such path must have a vertex projecting onto vertex B (see Figure 2.68). Thus there exist in C_2 edges a_1, a_2, and b; two of these must be oriented in opposite directions.

Case 1: The two oriented in opposite directions are a_1 and b, as shown in Figure 2.68. Without loss of generality, assume a_1 is above a_2. We investigate the path P from the "tip" of a_1 to the "tail" of b.

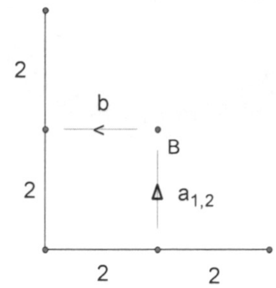

Figure 2.68

If the projection of P does not intersect C_1, then either C_2 has an irreducible crossing, or is reducible in length, or the crossings generated by a_1 and b with C_1 can be eliminated from the projection and do not contribute to the necessary linking between C_1 and C_2.

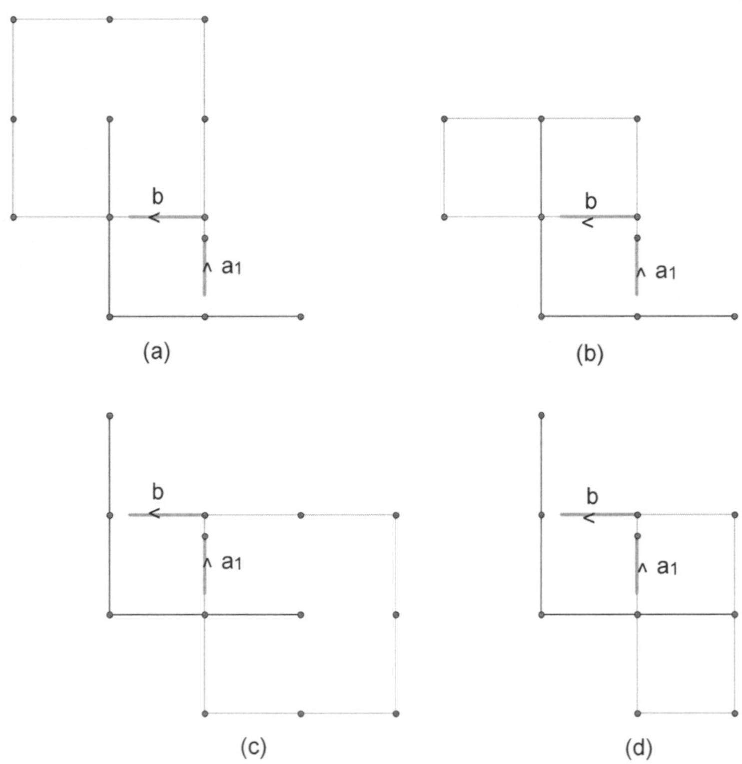

Figure 2.69

Thus P must introduce at least one intersection with C_1. If there is only one such intersection, then it will look like one of the loops in Figure 2.69, or an even bigger loop.

One can see that if such a loop is not reducible, then it must contribute to the linking with C_1. However, this cannot be; consider the path P as shown in Figure 2.69 (a). Let c be the x-step on P that projects to b. Consider the case where c is in the same direction as a_1, as shown in Figure 2.70 (a). If c is on a path that will pass through C_1 then the crossings generated by a_1 and c can be eliminated from the projection. If c will not pass through C_1, then C_2 is reducible in length.

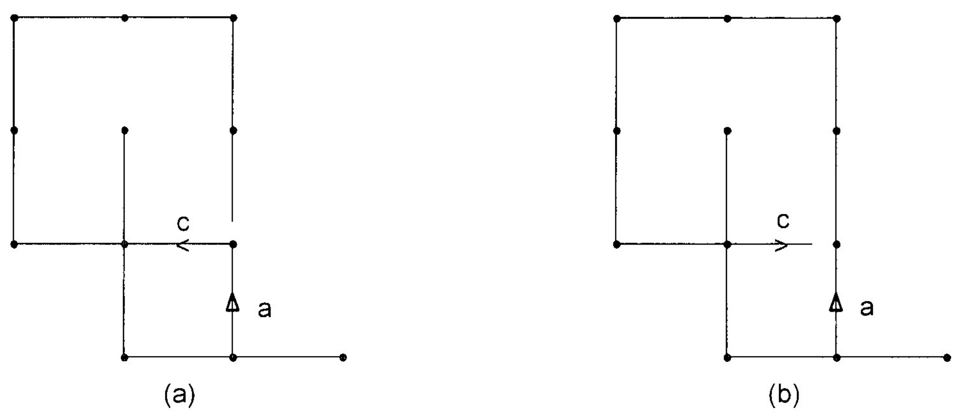

Figure 2.70

Consider c as shown in Figure 2.70 (b). In this case the passes through C_1 by c and a contribute +2 to the linking number. This requires two passes in opposite directions through C_1 to generate a linking number of zero. This will result in a C_2 that has more than 18 steps, the details are left to the reader.

Thus P must intersect C_1 twice in the projection in a non-reducible crossing. Then P could look as shown in Figure 2.71.

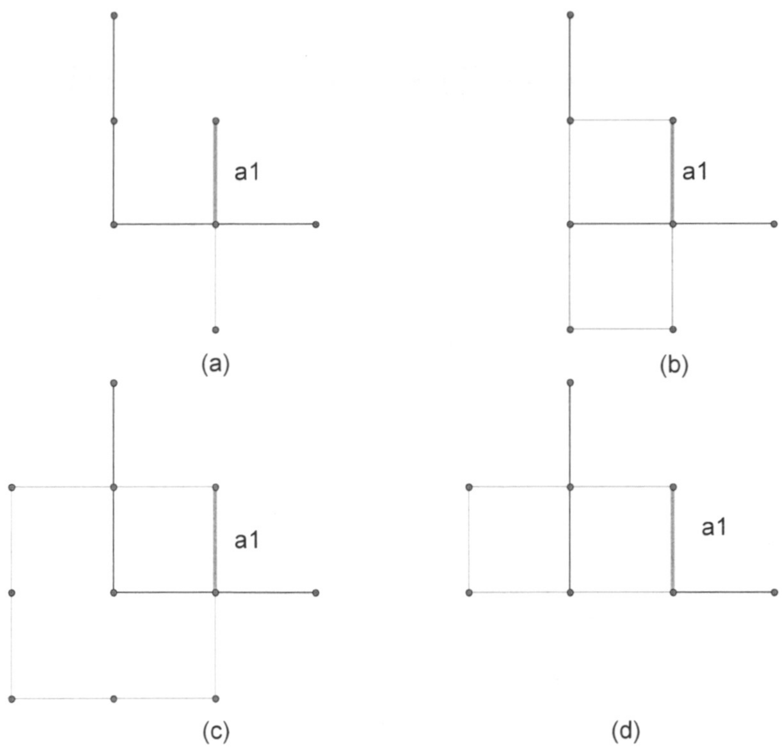

Figure 2.71

If P is not reducible, one of the passes must go "over" and the other "under" C_1. C_1 has a height differential of at least 2. Thus P must contain at least 6 z-steps which implies length(P) $\geq$ 10. Together with a_1 and b, this gives the length of C_2 so far as at least 12. The remaining 6 steps are not enough to connect the tail of a_1 to the tip of b so that the crossings introduced by a_1 and b with C_1 cannot be eliminated once we deform the link off lattice.

Case 2: Let a_1 and a_2 be the two that pass through C_1 in the opposite direction, as shown in Figure 2.72. Without loss of generality, a_1 is above a_2.

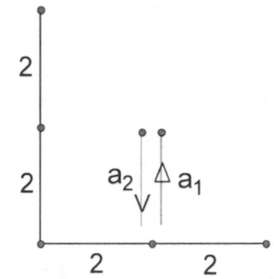

Figure 2.72

As before, we consider the path P_1 from the tip of a_1 to the tail of a_2. Let P_2 be the path from the tip of a_2 to the tail of a_1. If C_2 has 18 steps then either P_1 or P_2 must have a length of at most 8. For the same reason as in Case 1, this is impossible; the details are left to the reader.

Next we will consider the projection of C_1 as shown in Figure 2.67 (d). Up to symmetry, C_1 is as shown in Figure 2.73.

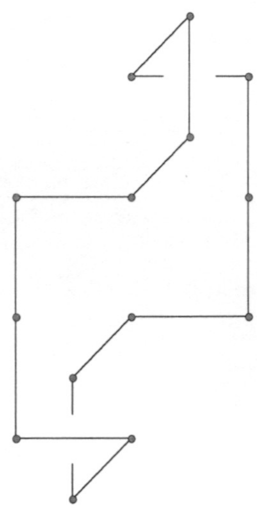

Figure 2.73

Note that C_1 does not bound a vertex, so such a C_1 cannot form any non-split link. All other cases in Figure 2.67 are left to the reader.

Case 4: C_1 has length 16

As we did in Case 3, it suffices to show we cannot form the Whitehead link on the cubic lattice with length(Whitehead link) ≤ 32. Thus we will show that when length(C_1) = 16, length(C_2) > 16. The following is another technical lemma. Note that Lemma 2.3 and its proof are very similar to those of Lemma 2.2.

Lemma 2.3: Let A and B be two lattice planes such that distance(A,B) = 2. No simple closed lattice curve C_2 of length 16 that intersects A to B by 4 paths can be part of a Whitehead lattice link L with length(L) < 34.

Proof of Lemma 2.3:

Let the projection of the planes be as shown in Figure 2.74.

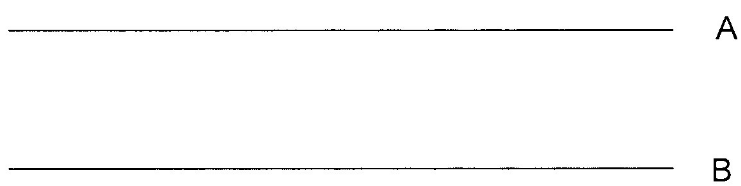

Figure 2.74

C_2 contains 4 paths of z-steps. Each path must have at least length 2; otherwise C_2 is reducible. If C_2 contains more than 8 z-steps, then C_2 must be reducible. The details are left to the reader.

Thus C_2 contains exactly 8 z-steps. The only way that C_2 is not immediately reducible is if the following occurs when we walk along C_2: 2 z-steps, 2 x- or y-steps, 2 z-steps, 2 x- or y-steps, 2 z-steps, 2 x- or y-steps, 2 z-steps, and 2 x- or y-steps. If any of the 2 x- or y-step pairs consist of one x- and one y-step, then C_2 cannot bound two vertices (details left to the reader). Thus C_2 must look like Figure 2.75. Note that no square move on C_2 is possible because all such moves make C_2 reducible.

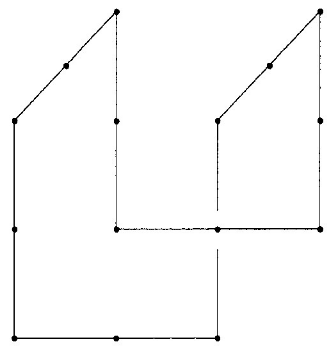

Figure 2.75

We now show that C_1 must have a length greater than 16. After an ε-deformation of C_2, we can project C_2 into the x-y plane as shown in Figure 2.76.

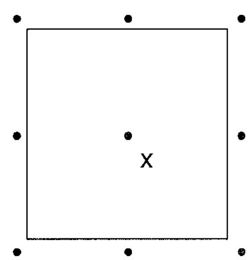

Figure 2.76

There have to be at least 6 intersections with C_2, which means the projection of C_1 passes through x at least 3 times. In the projection into the x-y plane it takes at least 8 steps for a non-reducible C_1 to return to x, so length(C_1) > 16. □

Without loss of generality, assume that the number of x-steps of C_1 is less than or equal to the number of y-steps of C_1, which is less than or equal to the number of z-steps of C_1. Then the possible combinations of C_1 are listed in Table 2.4.

x-steps	y-steps	z-steps
2	0	14
4	0	12
6	0	10
2	2	12
4	2	10
6	2	8
4	4	8
6	4	6

Table 2.4

Case 4.1: C_1 has length 16 and is planar

Similar to Case 3, Lemma 2.3 tells us we cannot form a minimal representation of the Whitehead link where C_1 has length 16 and is planar.

Case 4.2: C_1 has 2 x-steps, 2 y-steps and 12 z-steps

Figure 2.59 shows the possible projections up to symmetry onto the x-y plane where C_1 has 2 x-steps, 2 y-steps, and 12 z-steps. These cases are eliminated by the same reason as before (see Case 3.2).

Case 4.3: C_1 has 4 x-steps, 2 y-steps, and 10 z-steps

Figure 2.60 shows the possible projections onto the x-y plane (with respect to symmetry) of a C_1 with 4 x-steps, 2 y-steps, and 10 z-steps.

Consider the projection of C_1 as shown in Figure 2.60 (a). By Lemma 2.1, C_2 cannot have an irreducible crossing, thus our diagram must have at least 6 crossings.

Assume x_1 is not in the projection of C_2. In order to have at least 6 crossings, C_2 must have at least 3 paths (therefore at least 4 paths to maintain a linking number of zero) that connect plane A to plane B as shown in Figure 2.77. This is not possible by Lemma 2.3.

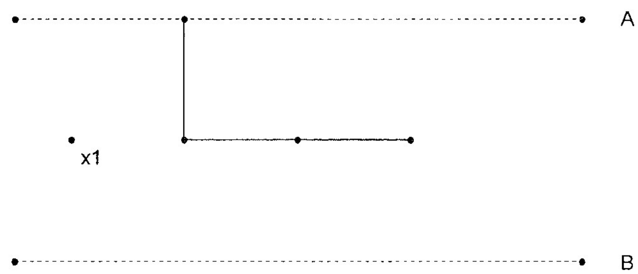

Figure 2.77

Now assume x_1 is in the projection of C_2. Let $B^* = B \cup \{x_1\}$. Let the 4 paths from A to B* be denoted P_1-P_4. Length(P_i) ≥ 2. If the projection of C_2 contains x_1, then there exists a path P_1 such that length(P_1) ≥ 3. Thus $\sum_{i=1}^{4} length(P_i) \geq 9$. Hence we only have 7 steps remaining to make 4 connections of the P_i's. This implies one such connection has length 1, thus it is reducible (details left to the reader). Therefore we cannot form the Whitehead link with such a C_1 and have length(Whitehead link) ≤ 32.

The cases of Figure 2.60 (b) - (d) are left to the reader.

Case 4.4: C_1 has 6 x-steps, 2 y-steps, and 8 z-steps

Figure 2.64 shows the possible projections onto the x-y plane (with respect to symmetry) of a C_1 with 6 x-steps, 2 y-steps, and 8 z-steps.

Similar to Case 4.3, no C_1 with a projection as shown in Figure 2.64 can be used to form the Whitehead link with length(Whitehead link) ≤ 32. Figure 2.78 shows the position of A and B and the x_1 for the projection of C_1 as shown in Figure 2.64 (a). The cases of Figure 2.64 (b) - (h) are left to the reader.

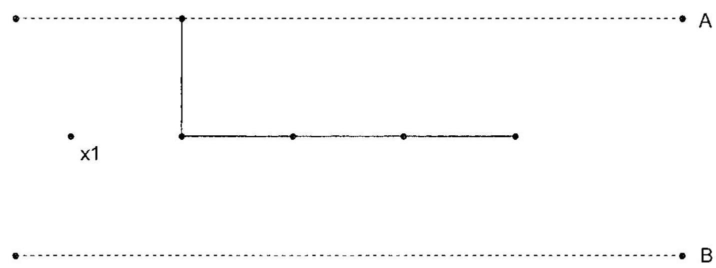

Figure 2.78

Case 4.5: C_1 has 4 x-steps, 4 y-steps, and 8 z-steps

Figure 2.67 shows the possible projections onto the x-y plane (with respect to symmetry) of a C_1 with 4 x-steps, 4 y-steps, and 8 z-steps. A projection of C_1 as shown in Figure 2.67 (a) is very similar to Case 3.5, the two added z-steps of C_1 do not allow us to form the Whitehead link such that length(C_2) ≤ 16 (details are left to the reader).

The two added z-steps of a C_1 whose projection is that of Figure 2.67 (d) could allow one strand to pass through C_1. However, we need at least two strands to pass through to form the Whitehead link, so this case is still impossible. The details of the other cases are left to the reader.

Case 4.6: C_1 has 6 x-steps, 4 y-steps, and 6 z-steps

There are many possibilities of projections of a C_1 with 6 x-steps, 4 y-steps, and 6 z-steps. Figure 2.79 shows the different projections (up to symmetry) that contain a degree eight or a degree six vertex. Note that a C_1 whose projection contains a degree eight vertex cannot be part of the Whitehead link (see Case 3.5 and Case 4.5).

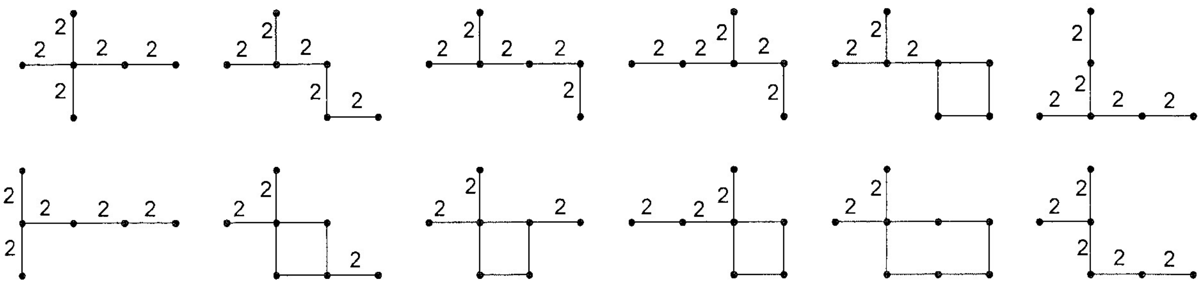

Figure 2.79

Figure 2.80 shows the different projections (up to symmetry) that contain a degree four vertex.

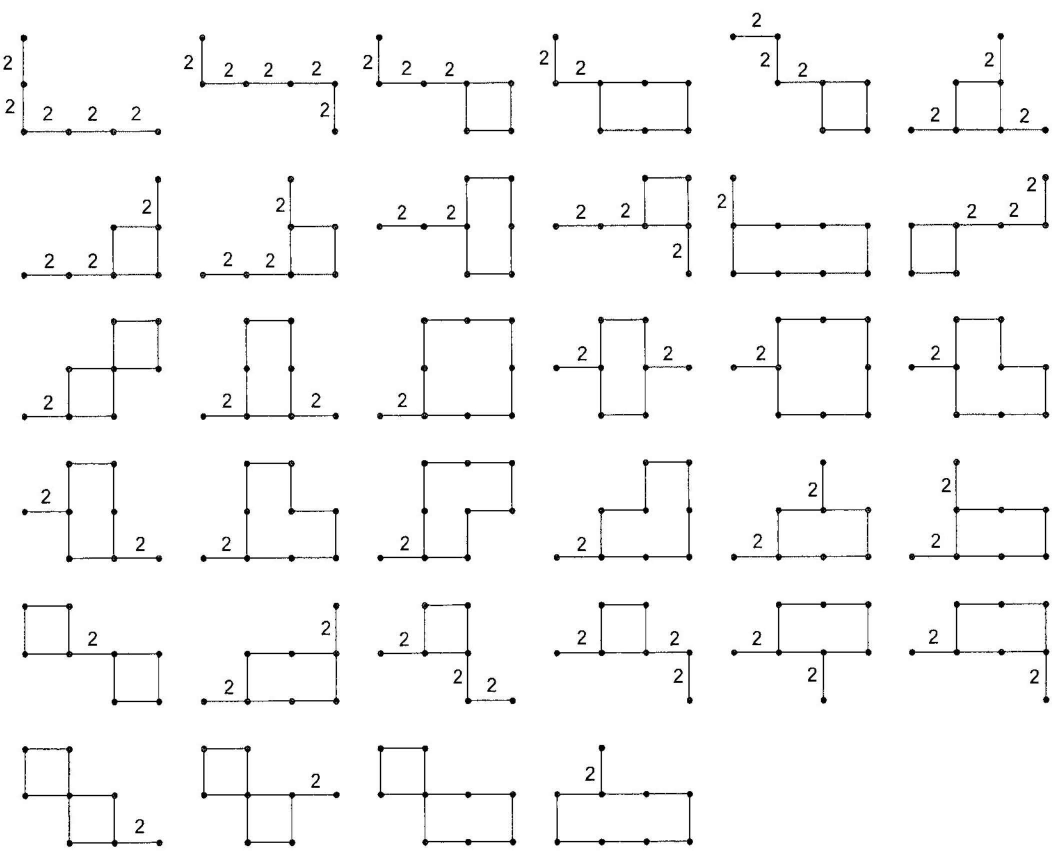

Figure 2.80

Figure 2.81 shows the different projections (up to symmetry) that contain only degree two vertices.

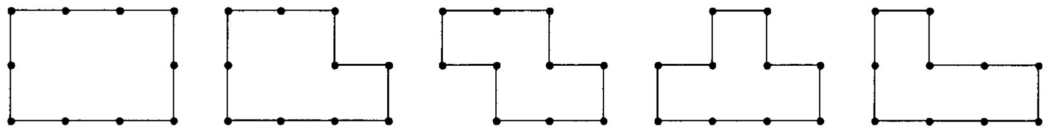

Figure 2.81

Case 4.6 has more possibilities of projections than cases 4.1-4.5. However our previous work will allow us to prove another lemma that will help eliminate cases as shown in Figures 2.79 – 2.81. We know now that the Whitehead link cannot be realized with a total length of at most 32 when one component has a length of less than 16 steps. Assume that we are given a 32-step representation of the Whitehead link with two components C_1 and C_2, each of length 16. Let x_i be the number of x-steps in C_i ($i = 1,2$). Let y_i be the number of y-steps in C_i ($i = 1,2$). Let z_i be the number of z-steps in C_i ($i = 1,2$).

Lemma 2.4: $\text{Maximum}\{x_1, x_2, y_1, y_2, z_1, z_2\} = 6$

Proof of Lemma 2.4: Given C_i we have $x_i + y_i + z_i = 16$. If one of x_i, y_i, or z_i is greater than six, then C_i has a projection onto the x-y, y-z, or x-z plane with at most 8 steps, so this is not possible (our previous work in Cases 3 and 4 proves this). If all x_i, y_i, and z_i are less than six, then $x_i + y_i + z_i$ is less than 16, which is impossible. □

We can now use this as follows: Given a projected C_1 then all we have to show is that a C_2 must use more than 6 x-, y-, or z-steps and we are done (we will leave most cases to the reader).

For example, consider the projection of C_1 as shown in Figure 2.82. As before, we know we must have at least 3 paths passing through or "above"/"below" C_1 so that neither component has an irreducible crossing.

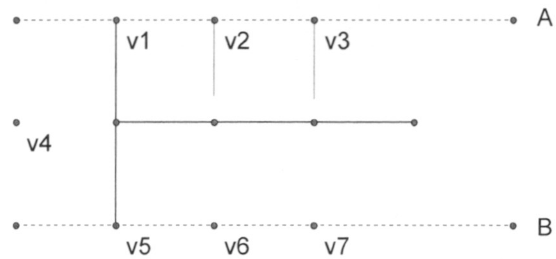

Figure 2.82

From Lemma 2.3 or Lemma 2.4, we know such a component cannot form the Whitehead link with at most 32 steps if each path projects to one of the line segments $\overline{v_1 v_5}$, $\overline{v_2 v_6}$, or $\overline{v_3 v_7}$.

Assume each of the three paths' (a_1, a_2, and b) projection includes vertex v_4. In order for C_1 to not be immediately reducible, C_2's projection must include vertex v_2 and v_3. Thus C_2 uses at least 6 x-steps. We cannot connect these 3 paths without using any more x-steps. So by Lemma 2.4, this projection cannot form the Whitehead link with at most 32 steps. By examining each combination of projections of a_1, a_2, and b and the paths that connect them, we know a C_1 with such a projection cannot yield a 32-step representation of the Whitehead link.

Now consider C_1 as shown in Figure 2.83.

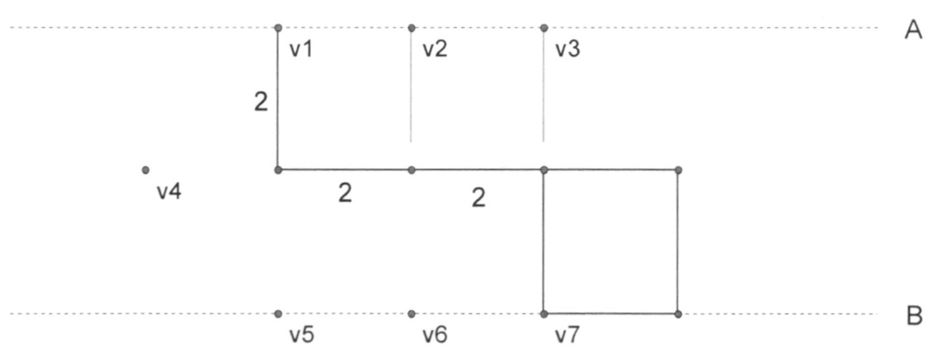

Figure 2.83

From Lemma 2.3 or Lemma 2.4, we know such a component cannot form the Whitehead link with at most 32 steps if each path projects to one of the line segments $\overline{v_1 v_5}$, $\overline{v_2 v_6}$, or $\overline{v_3 v_7}$.

Assume each of the three paths' (a_1, a_2, and b) projection includes vertex v_4. In order for C_1 to not be immediately reducible, C_2's projection must include vertex v_2 and v_3. Thus C_2 uses at least 6 x-steps. We cannot connect these 3 paths without using any more x-steps. So by Lemma 2.4, this projection cannot form the Whitehead link with at most 32 steps. By examining each combination of projections of a_1, a_2, and b and the paths that connect them, we know a C_1 with such a projection cannot yield a 32-step representation of the Whitehead link. All other cases are left to the reader.

This concludes the proof of Theorem 2.1. We know we can form an embedding on the cubic lattice of the Whitehead link with 34 steps when one component has length 12. We may or may not be able to construct a 34 step Whitehead link on the cubic lattice when one component has length 14 or 16. Most importantly, the Whitehead link cannot be embedded on the cubic lattice with 32 steps no matter the size of either component. Thus we know the Whitehead link cannot be realized on the cubic lattice with less than 34 steps.

References

[1] Stahl, S. (2005) *Introduction to Topology and Geometry*, Hoboken, New Jersey: Wiley-Interscience.

[2] Livingston, C. (1993) *Knot Theory*, Washington, DC: The Mathematical Association of America.

[3] Kinsey, L. C. (1993) *Topology of Surfaces*, New York: Springer-Verlag.

[4] Lickorish, W.B.R. (1997) *An Introduction to Knot Theory*, New York: Springer-Verlag.